REPRESENTATION OF FUNCTIONS AS A SUPERPOSITION OF WAVES IN TWO-DIMENSIONAL AND THREE-DIMENSIONAL LOBACHEVSKY SPACES

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Abstract

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REPRESENTATION OF FUNCTIONS AS A SUPERPOSITION OF WAVES IN TWO-DIMENSIONAL AND THREE-DIMENSIONAL LOBACHEVSKY SPACES

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The article develops a method for expanding an arbitrary function, given on a pseudosphere, in spherical waves. This method proves to be the most convenient in solving the Cauchy problem on the evolution of arbitrary small initial inhomogeneities in the open Friedmann model. With the aid of the method of spherical waves, an inversion formula is found for the expansion in plane waves (the direct Fourier transform), without the elegant but complicated apparatus of integral geometry on homogeneous spaces developed by I. M. Gelfand and M. I. Graev ([1]). On the two-dimensional pseudosphere, formulas have been obtained for expanding functions in cylindrical waves, and a construction based on them is given for inversion formulas for the expansion in plane waves. Functions depending only on the distance to a certain point in 3-dimensional Lobachevsky space were first expanded in spherical waves by I. S. Shapiro ([2]). A complete orthogonal system on the pseudosphere was first constructed by E. M. Lifshitz without inversion formulas ([3]). N. Ya. Vilenkin and Ya. A. Smorodinsky considered the problem of expansion in series in eigenfunctions of the Laplace operator on the 3-dimensional pseudosphere in various orthogonal coordinate systems and found inversion formulas for the expansions in series in these systems ([4]).

The method of expansion of an arbitrary function in plane waves (the Fourier integral) on the hyperplane $x_0 = \text{const}$ and on the pseudosphere $x_0^2 - x_1^2 - x_2^2 - x_3^2 = a^2$ can be approached from a unified point of view—the solution of the Cauchy problem for the wave equation in Minkowski space:

$$\partial^2\varphi/\partial x_0^2 - \partial^2\varphi/\partial x_1^2 - \partial^2\varphi/\partial x_2^2 - \partial^2\varphi/\partial x_3^2 = 0 \tag{1}$$

with initial data on the above-mentioned hypersurfaces, since constructing the general solution of the wave equation both on the plane and on the pseudosphere is equivalent to finding a complete integral (a plane wave) that is an eigenfunction of the Laplace operator on these manifolds.

In order to construct the Fourier integral for functions given on the pseudosphere, in the wave equation (1) we pass to the coordinates

$$\tau = \sqrt{x_0^2 - x_1^2 - x_2^2 - x_3^2}, \qquad y_\alpha = x_\alpha/\tau \quad (\alpha = 1, 2, 3) :$$

$$\tau^{-3}\frac{\partial}{\partial\tau}\left[\tau^3\frac{\partial\varphi}{\partial\tau}\right] = \frac{\sqrt{1+y^2}}{\tau^2}\frac{\partial}{\partial y_\alpha}\left[\frac{\delta_\alpha^\beta + y_\alpha y_\beta}{\sqrt{1+y^2}}\frac{\partial}{\partial y_\beta}\right]^* . \qquad (1')$$

The complete integral of the wave equation $(1')$ has the form $A(\tau(\sqrt{1+y^2}\xi_0 - \xi_1 y_1 - \xi_2 y_2 - \xi_3 y_3))^{-1+i\alpha}$, where the form of the exponent $-1+i\alpha$ is determined by the boundedness requirement as $\tau \to \infty$. The vector $(\xi_0, \xi_1, \xi_2, \xi_3)$ is isotropic, $\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0$. The eigenfunction of the Laplace operator in Lobachevsky space $[y, \xi]^{-1+i\alpha} = (\sqrt{1+y^2}\xi_0 - \xi_1 y_1 -$

* Here and below $y^2 = y_1^2 + y_2^2 + y_3^2$.

$-\xi_2 y_2 - \xi_3 y_3)^{-1+i\alpha}$ is entirely analogous to the plane wave $\exp(ikx)$, while the so-called orisphere $[y, \xi] =$ const plays the role of a plane in Euclidean space $((^1), p.372)$. Therefore the general integral of equation $(1')$ is written in the form

$$\varphi(\tau, y) = \int_0^\infty d\alpha \int d\xi\, A(\xi, \alpha)(\tau[y, \xi])^{-1+i\alpha}, \qquad (2)$$

where $d\xi$ is the measure on the cone $((^1), p.381)$, and the function $A(\xi, \alpha)$ must be homogeneous of degree $-1 - i\alpha$. Using the homogeneity property, the general integral (2) can also be rewritten in the form

$$\varphi(\tau, y) = \int_0^\infty d\alpha \int_\Gamma d\Gamma\, B(\xi, \alpha)(\tau[y, \xi])^{-1+i\alpha}, \qquad (2')$$

where Γ is an arbitrary closed "contour" on the cone, intersected once by a generatrix, and $d\Gamma$ is the invariant measure of the contour $((^1), p.383)$. Expression $(2')$ gives an expansion of the solution of the wave equation $(1')$ in plane waves in Lobachevsky space.

In order to be able to solve the Cauchy problem with initial data at $\tau = a =$ const, it is necessary to construct the inversion formula for $(2')$. For this we use the method of spherical means. Denote by $\varphi_\mu(y)$ the mean value of $\varphi(y)$ over the sphere of radius μ with center at the point y_0:

$$\varphi_\mu(y_0) = \left(4\pi a^2 \operatorname{sh}^2 \frac{\mu}{a}\right)^{-1} \oint\limits_{[y,y_0]=\operatorname{ch}\mu/a} \varphi(y)\, ds \equiv$$

$$\equiv (4\pi)^{-1} \int \varphi(y_\alpha(y_0,\theta,\varphi,\mu)) \sin\theta\, d\theta\, d\varphi,$$

$$y_\alpha(y_0,\theta,\varphi,\mu) = \frac{y_{\alpha 0}(y_{10}\omega_1 + y_{20}\omega_2 + y_{30}\omega_3)}{y_{10}^2 + y_{20}^2 + y_{30}^2} \left(\sqrt{1+y_0^2} - 1\right) \operatorname{sh}\frac{\mu}{a} +$$

$$+ y_\alpha \operatorname{ch}\frac{\mu}{a} + \omega_\alpha \operatorname{sh}\frac{\mu}{a},$$

where $\omega_1 = \cos\theta,\ \omega_2 = \sin\theta\sin\varphi,\ \omega_3 = \sin\theta\cos\varphi$.

For the spherical means $\varphi_\mu(y)$ in three-dimensional Lobachevsky space there is the curious formula

$$\Delta\varphi_\mu(y) = \frac{\sqrt{1+y^2}}{a^2}\frac{\partial}{\partial y_\alpha}\left[\frac{\delta_\alpha^\beta + y_\alpha y_\beta}{\sqrt{1+y^2}}\frac{\partial}{\partial y_\beta}\varphi_\mu(y)\right] = \operatorname{sh}^{-2}\frac{\mu}{a}\frac{\partial}{\partial\mu}\left[\operatorname{sh}^2\frac{\mu}{a}\frac{\partial}{\partial\mu}\varphi_\mu(y)\right], \tag{3}$$

proved by direct differentiation. From (3) it follows that the eigenfunctions for the Laplace operator in the class of spherical means $\{\varphi_\mu(y)\}$ are the functions

$$\frac{\sin(\omega\mu/a)}{\operatorname{sh}(\mu/a)}$$

and only these. Consequently, the arithmetic mean of an arbitrary function $\varphi(y)$ over the sphere μ admits the integral expansion

$$\varphi_\mu(y) = \int_0^\infty \varphi(\omega, y)\frac{\sin(\omega\mu/a)}{\omega\ \operatorname{sh}(\mu/a)}\, d\omega, \tag{4}$$

the coordinates y_α entering this expansion as parameters.

The inversion formula for (4) has the form:

$$\varphi(\omega, y_0) = \frac{\omega}{2\pi a}\int_0^\infty \sin\left(\omega\frac{\mu}{a}\right)\operatorname{sh}\frac{\mu}{a}\,\varphi_\mu(y_0)\, d\mu = \frac{\omega}{4\pi^2}\int \frac{\sin(\omega r(y,y_0)/a)}{\operatorname{sh}(r(y,y_0)/a)}\varphi(y)\,\frac{dy_1 dy_2 dy_3}{\sqrt{1+y^2}} \tag{5}$$

(this is proved with the aid of the one-sided Fourier transform). From formula (5) it follows that

$$\int_0^\infty \varphi(\omega, y_0)\, d\omega = \int_0^\infty \delta'(\mu)\,\mathrm{sh}\,\mu\, \varphi_\mu(y_0)\, d\mu = \varphi_{\mu=0}(y_0) = \varphi(y_0). \qquad (6)$$

Formulas (5), (6) give an expansion of an arbitrary function in pseudospherical waves
$\dfrac{\sin(\omega r(y, y_0)/a)}{\mathrm{sh}(r(y, y_0)/a)}$, where $r(y, y_0) = a\ \mathrm{ar\,ch}[y, y_0]$.

The corresponding expansions in spherical waves in Euclidean space, as is known, have the form

$$\varphi(x) = \int_0^\infty \varphi(\omega, x)\, d\omega; \qquad \varphi(\omega, x) = \int \frac{\sin \omega r(x, x')}{r(x, x')}\, \varphi(x')\, dx', \qquad (5')$$

where $r^2(x, x') = (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2$.

Let us take the arithmetic mean of both sides in $(2')$ over the sphere of radius μ with center at y_0 for $\tau = a$ (we include the factor $a^{-1+i\alpha}$ in $B(\xi, \alpha)$). By the formula

$$(4\pi)^{-1} \int_{[y, y_0] = \mathrm{ch}\,\mu/a} [y, \xi]^{-1+i\alpha} \sin\theta\, d\theta\, d\varphi = [y_0, \xi]^{-1+i\alpha} \frac{\sin(\alpha\mu/a)}{\alpha\ \mathrm{sh}(\mu/a)}$$

from $(2')$ we obtain

$$\varphi_\mu(y_0) = \int_0^\infty d\alpha \left\{ \int_\Gamma B(\xi, \alpha)\, \frac{\sin(\alpha\mu/a)}{\alpha\ \mathrm{sh}(\mu/a)} [y_0, \xi]^{-1+i\alpha}\, d\Gamma \right\}. \qquad (7)$$

From comparison of (7) and (4) it follows that α/a has the meaning of the wave number ω/a and that

$$\int_\Gamma B(\xi, \omega)[y_0, \xi]^{-1+i\omega}\, d\Gamma = \varphi(\omega, y_0) = \frac{\omega}{4\pi^2} \int \frac{\sin(\omega r(y, y_0)/a)}{\mathrm{sh}\ r(y, y_0)/a}\, \varphi(y)\, dy. \qquad (8)$$

The pseudospherical wave $\sin(\omega r(y, y_0)/a)/\,\mathrm{sh}(r(y, y_0)/a)$ can be represented as a superposition of plane waves with the same wave number:

$$\frac{\sin(\omega r(y, y_0)/a)}{\mathrm{sh}(r(y, y_0)/a)} = \frac{\omega}{4\pi} \int_\Gamma [y_0, \xi]^{-1+i\omega} [y, \xi]^{-1-i\omega}\, d\Gamma. \qquad (9)$$

From (8) and (9) we have

$$\int_\Gamma [y_0,\xi]^{-1+i\omega}\, d\Gamma \left\{ B(\xi,\omega) - \frac{\omega^2}{16\pi^3} \int \varphi(y)[y,\xi]^{-1-i\omega}\, dy \right\} = 0. \tag{10}$$

The obtained integral equation (10) has the trivial solution

$$B(\xi,\omega) = \frac{\omega^2}{16\pi^3} \int \varphi(y)[y,\xi]^{-1-i\omega}\, dy.$$

If one specifies the contour Γ and expands the function $B(\xi,\omega)$ on it in a series, then with the aid of the inversion formula one can obtain the known inversion formulas of N. Ya. Vilenkin and Ya. A. Smorodinskii (4).

Let us repeat analogous arguments in the two-dimensional Lobachevsky space.

1) For the mean (cylindrical mean) $\psi_\mu(y_0)$ over a circle of radius μ with center at the point y_0 of the function $\psi(y)$, the formula holds:

$$\frac{\sqrt{1+y^2}}{a^2} \frac{\partial}{\partial y_\alpha} \left[\frac{\delta^\alpha_\beta + y_\alpha y_\beta}{\sqrt{1+y^2}} \frac{\partial}{\partial y_\beta} \psi_\mu(y) \right] = \left(\operatorname{sh} \frac{\mu}{a} \right)^{-1} \left[\frac{\partial}{\partial \mu} \operatorname{sh} \frac{\mu}{a} \frac{\partial}{\partial \mu} \psi_\mu(y) \right], \tag{11}$$

where α, β take the values $1, 2$; $y^2 = y_1^2 + y_2^2$.

2) It follows from (11) that the function $\psi_\mu(y)$ is expanded in the functions $P_{-1/2+i\omega}\left(\operatorname{ch} \frac{\mu}{a}\right)$—the eigenfunctions of the operator $\left(\operatorname{sh}\frac{\mu}{a}\right)^{-1} \frac{\partial}{\partial\mu} \left[\operatorname{sh}\frac{\mu}{a} \frac{\partial}{\partial\mu}\right]$:

$$\psi_\mu(y) = \int_0^\infty \psi(\omega, y)\, P_{-1/2+i\omega} \left(\operatorname{ch} \frac{\mu}{a} \right) d\omega, \tag{12}$$

where $P_{-1/2+i\omega}(z)$ is the Legendre function of the first kind (conical function).

3) Let us solve relation (12) for $\psi(\omega, y)$, using the orthogonality property of the conical functions:

$$\int_1^\infty P_{-1/2+i\omega}(z) P_{-1/2+i\omega'}(z)\, dz = \delta(\omega-\omega')\, \omega \operatorname{th} \pi\omega,$$

$$\psi(\omega, y) = (2\pi)^{-1} \int P_{-1/2+i\omega} \left(\operatorname{ch} \frac{r(y,y')}{a} \right) \omega \operatorname{th} \pi\omega\, \psi(y')\, dy'. \tag{13}$$

From formula (13) it follows that

$$\psi(y) = \int_0^\infty \psi(\omega, y)\, d\omega, \tag{14}$$

since $\psi_{\mu=0}(y) = \psi(y)$.

Formulas (13), (14) give the expansion of an arbitrary function in the two-dimensional Lobachevsky space into cylindrical waves.

With the aid of the method of cylindrical waves, analogously to the three-dimensional case, one can find the transform inverse to the direct Fourier transform

$$\psi(y) = \int_\Gamma d\Gamma \int_0^\infty d\alpha \, [y,\xi]^{-1/2+i\alpha} B(\xi,\alpha)$$

(where Γ is a contour on the two-dimensional cone $\xi_0^2 - \xi_1^2 - \xi_2^2 = 0$):

$$B(\xi,\alpha) = \frac{\alpha \, \mathrm{th}\, \pi\alpha}{4\pi^2} \int [y,\xi]^{-1/2-i\alpha} \psi(y) \, dy.$$

Here it is necessary to use the formulas

$$(2\pi)^{-1} \int_{[y,y_0]=\mathrm{ch}\, \mu/a} [y,\xi]^{-1/2+i\alpha} \, d\theta = [y_0,\xi]^{-1/2+i\alpha} P_{-1/2+i\alpha}\left(\mathrm{ch}\, \frac{\mu}{a}\right);$$

$$2\pi P_{-1/2+i\omega}([y,y_0]) = \int_\Gamma [y,\xi]^{-1/2-i\omega} [y_0,\xi]^{-1/2-i\omega} \, d\Gamma.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.