DIFFRACTION OF ELASTIC WAVES

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Abstract

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THEORY OF ELASTICITY

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DIFFRACTION OF ELASTIC WAVES BY AN INFINITE ROW OF CIRCULAR CYLINDERS

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Static periodic problems of the plane theory of elasticity have been considered by a number of authors ($^{1-5,7,8}$). In most works these problems are reduced to infinite systems of algebraic equations. In the present paper, three-dimensional problems of diffraction of elastic harmonic waves by an infinite row of identical circular cylinders with parallel axes are solved.

Let the region occupied by the elastic body be a space with an infinite number of cylindrical cavities of radius R. Introduce cylindrical coordinate systems (r_k, φ_k, z_k), whose z_k-axes coincide with the longitudinal axes of the cavities, and the planes $z_k = 0$ coincide $(k = 0, \pm 1, \pm 2, ...)$. Thus, the plane $z_k = 0$ contains an infinite row of identical circular holes, whose centers lie on the x-axis (the x_k-axes also coincide). Denote by δ the distance between the origins of two neighboring coordinate systems.

As is known, the equations of motion of an elastic body in cylindrical coordinates, by introducing wave potentials, can be represented in the form

$$c_1^2 \Delta \Phi^{(*)} - \partial^2 \Phi^{(*)}/\partial t^2 = 0, \qquad c_2^2 \Delta \Psi_j^{(*)} - \partial^2 \Psi_j^{(*)}/\partial t^2 = 0 \quad (j = 1, 2), \tag{1}$$

where c_1 and c_2 are the propagation velocities of longitudinal and transverse waves. The components of the displacement vector (u_r, u_φ, u_z) are expressed in terms of the functions $\Phi^{(*)}$ and $\Psi_j^{(*)}$ as follows:

$$u_r = \frac{\partial \Phi^{(*)}}{\partial r} + \frac{1}{r}\frac{\partial \Psi_1^{(*)}}{\partial \varphi} + \frac{\partial^2 \Psi_2^{(*)}}{\partial r\,\partial z},$$

$$u_\varphi = \frac{1}{r}\frac{\partial \Phi^{(*)}}{\partial \varphi} - \frac{\partial \Psi_1^{(*)}}{\partial r} + \frac{1}{r}\frac{\partial^2 \Psi_2^{(*)}}{\partial \varphi\,\partial z}, \tag{2}$$

$$u_z = \frac{\partial \Phi^{(*)}}{\partial z} - \frac{1}{r^2}\frac{\partial^2 \Psi_2^{(*)}}{\partial \varphi^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi_2^{(*)}}{\partial r}\right).$$

Suppose that in the given body, in the direction l, forming an angle γ_2 with the oz-axis, there propagates a plane harmonic wave of longitudinal type

$$\Phi_0^{(*)} = \exp[i\alpha_1 r \sin\gamma_2 \cos(\varphi - \gamma_1)] \exp[i(\alpha_1 z \cos\gamma_2 - \omega t)], \tag{3}$$

where γ_1 is the angle between the planes xz and lz, and $\alpha_1 = \omega/c_1$.

Taking into account the periodicity of $\Phi_0^{(*)}$ with respect to z, represent the potentials $\Phi^{(*)}$ and $\Psi_j^{(*)}$ in the form

$$\Phi^{(*)} = \Phi(r,\varphi) \exp[i(\alpha_1 z \cos\gamma_2 - \omega t)],$$

$$\Psi_j^{(*)} = \Psi_j(r,\varphi) \exp[i(\alpha_1 z \cos\gamma_2 - \omega t)],$$

after which the wave equations (1) are transformed into equations of oscillations

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \alpha^2 = 0,$$

$$\frac{\partial^2 \Psi_j}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi_j}{\partial \varphi^2} + \beta^2 \Psi_j = 0 \quad (j = 1, 2) \tag{4}$$

$$\left(\alpha^2 = \frac{\omega^2}{c_1^2}\sin^2\gamma_2, \quad \beta^2 = \frac{\omega^2}{c_2^2} - \frac{\omega^2}{c_1^2}\cos^2\gamma_2\right).$$

For the time being we shall assume that $\gamma_2 \neq 0,\ \pi/2$.
We write the boundary conditions on the lateral surfaces of the cavities in the form

$$\left[D_2^{(k)}(\Phi) - \frac{1}{r_k}\frac{\partial}{\partial \varphi_k} D_1^{(k)}\left(\Psi_1 - ia_1\cos\gamma_2 \frac{\partial^2 \Psi_2}{\partial r_k^2}\right)\right]_{r_k=R} = 0,$$

$$\left[-\frac{1}{r_k}\frac{\partial}{\partial \varphi_k} D_1^{(k)}(\Phi) + D_2^{(k)}(\Psi_1) + (1-b)a_1^2\cos^2\gamma_2 \Psi_1 - ia_1\cos\gamma_2 \frac{1}{r_k}\frac{\partial}{\partial \varphi_k} D_1^{(k)}(\Psi_2)\right]_{r_k=R} = 0, \tag{5}$$

$$\left[ia_1\cos\gamma_2\frac{\partial\Phi}{\partial r_k} + ia_1\cos\gamma_2(1-b)\frac{1}{r_k}\frac{\partial\Psi_1}{\partial\varphi_k} + (b\beta^2 - a_1^2\cos^2\gamma_2)\frac{\partial\Psi_2}{\partial r_k}\right]_{r_k=R} = 0,$$

where

$$D_1^{(k)}(f) = \frac{1}{r_k}f - \frac{\partial f}{\partial r_k}, \qquad D_2^{(k)}(f) = \xi f + \frac{1}{r_k}\frac{\partial f}{\partial r_k} + \frac{1}{r_k^2}\frac{\partial^2 f}{\partial\varphi_k^2},$$

$$\xi = \begin{cases}\alpha_1^2(a\sin^2\gamma_2 - \cos^2\gamma_2), & f = \Phi,\\ \alpha_1^2(a - \frac{3}{2}\cos^2\gamma_2), & f = \Psi.\end{cases}$$

If the surfaces S_k are rigidly fixed, then in (5) $a = b = 0$; if S_k are free of stresses, then $a = (\lambda + 2\mu)/2\mu,\ b = 1/2$.

Let us denote the potential of the incident wave in the k-th coordinate system by $\Phi_0^{(k)}(x_k, y_k)$. Then, as is easy to see,

$$\Phi_0^{(q)}(x_q, y_q) = \exp(iq\gamma)\Phi_0^{(0)}(x_q, y_q) \quad (\gamma = a_1\delta\sin\gamma_2\cos\gamma_1). \tag{6}$$

If the potentials of the reflected waves Φ_1 and Ψ_j are chosen so that for them relation (6) is satisfied, then it will be sufficient to satisfy the boundary conditions only on the surface S_0. On the other surfaces they will be satisfied automatically. This is achieved by the representation

$$\Phi_1 = \sum_n\sum_k A_n\exp(ik\gamma)H_p(\alpha r_k)\exp(in\varphi_k),$$

$$\Psi_j = \sum_n\sum_k B_n^{(j)}\exp(ik\gamma)H_n(\beta r_k)\exp(in\varphi_k). \tag{7}$$

In expressions (7) the summation indices range from $-\infty$ to $+\infty$, and H_n is the Hankel function of the first kind.

Using the addition theorem for cylindrical functions (6), we write the potentials in the coordinate system (r_0, φ_0):

$$\Phi_0^{(0)}(r_0, \varphi_0) = \sum_n i^n J_n(\alpha r_0)\exp[in(\varphi_0 - \gamma_1)],$$

$$\Phi_1(r_0, \varphi_0) = \sum_n [A_{nH}n(\alpha r_0) + S_{nJ}n(\alpha r_0)]\exp(in\varphi_0), \tag{8}$$

$$\Psi_j(r_0,\varphi_0)=\sum_n\left[B_n^{(j)}H_n(\beta r_0)+Q_n^{(j)}J_n(\beta r_0)\right]\exp(in\varphi_0)\quad(r_0<\delta),$$

where

$$S_n=\sum_p\sum_{k=1}^{\infty}A_p\left[\exp(ik\gamma)H_{n-p}(\alpha k\delta)+\exp(-ik\gamma)H_{p-n}(\alpha k\delta)\right],$$

$$Q_n^{(j)}=\sum_p{}'\sum_{k=1}^{\infty}B_p^{(j)}\{\exp(ik\gamma)H_{n-p}(\beta k\delta)+\exp(-ik\gamma)H_{p-n}(\beta k\delta)\}.$$

Satisfying the boundary conditions (5) for $k=0$, we obtain an infinite system of algebraic equations

$$A_n\xi_{n,j}+B_n^{(1)}\eta_{n,j}+B_n^{(2)}\zeta_{n,j}=$$
$$=-\xi_{n,j+3}S_n-\eta_{n,j+3}Q_n^{(1)}-\zeta_{n,j+3}Q_n^{(2)}-i^n\exp(-in\gamma_1)\xi_{n,j+3}\qquad(9)$$

$$(j=1,2,3;\quad n=0,\pm1,\pm2,...).$$

If in system (9) we make a change of unknowns, putting

$$X_{n,j}=A_n\xi_{n,j}+B_n^{(1)}\eta_{n,j}+B_n^{(2)}\zeta_{n,j},$$

then it assumes the canonical form

$$X_{n,j}=\sum_p[\chi_{n,p,j}X_{p,1}+\gamma_{n,p,j}X_{p,2}+\delta_{n,p,j}X_{p,3}]-i^n\exp(-in\gamma_1)\xi_{n,j+3}\qquad(10)$$

$$(j=1,2,3;\quad n=0,\pm1,\pm2,...).$$

The infinite system (10), provided the conditions

$$\alpha_1\delta\sin\gamma_2(1\pm\cos\gamma_1)\neq m_1\cdot2\pi,\qquad\delta(\beta\pm\alpha_1\cos\gamma_1\sin\gamma_2)\neq m_2\cdot2\pi,\qquad(11)$$

where m_1 and m_2 are integers, has a determinant of normal type. This follows from the convergence of the double series composed of the moduli of its coefficients. In proving the convergence of this series, asymptotic formulas for cylindrical functions with large index are used, as well as inequalities of the type

$$\left|\sum_{k=1}^{\infty} \exp(ik\gamma) H_{m-p}(\alpha k\delta)\right| < M[\,|R_{|p-n|,0}(i\alpha\delta)| + |R_{|p-n|-1,1}(i\alpha\delta)|\,],$$

where R are Lommel polynomials (^{6}), and M is a constant.

Thus, the infinite system (10) under conditions (11) has a unique bounded solution if its determinant is nonzero. This solution can be found approximately by the reduction method.

Remark. Above it was assumed that $\gamma_2 \neq 0; \pi/2$. If $\gamma_2 = \pi/2$, then the three-dimensional problem becomes a two-dimensional one. In this case one should take $\Psi_2 = 0$. If $\gamma_2 = 0$, then the first equation in (4) must be replaced by Laplace' s equation $\Delta\Phi = 0$, whose solution has the form

$$\Phi = B\left[\ln\zeta\cdot\bar{\zeta} + \sum_{k=1}^{\infty} \ln\left(1 - \frac{\zeta^2}{k^2\delta^2}\right)\left(1 - \frac{\bar{\zeta}^2}{k^2\delta^2}\right)\right] + \varphi(\zeta) + \overline{\psi(\zeta)},$$

$$\varphi(\zeta) = \sum_{p=1}^{\infty}\sum_{k} \frac{\alpha_p}{(\zeta - k\delta)^p}, \qquad \psi(\zeta) = \sum_{p=1}^{\infty}\sum_{k} \frac{\alpha_p^{(*)}}{(\zeta - k\delta)^p}, \qquad \zeta = r\exp(i\varphi).$$

In both cases, as a result of satisfying the boundary conditions one obtains an infinite system of algebraic equations with a determinant of normal type.

The restrictions (11) have a definite physical meaning. Thus, for $\gamma_1 = \gamma_2 = \pi/2$ they express the fact that the distance between the centers of the cavities must not be a multiple either of the wavelength of the longitudinal type or of the wavelength of the transverse type.

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Note: Figure translations are in progress. See original paper for figures.

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