Reports of the Academy of Sciences of the USSR

MATHEMATICAL PHYSICS

1969

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1969, Volume 184, No. 5

UDC 532.517.4

MATHEMATICAL PHYSICS

E. A. NOVIKOV

SCALE SIMILARITY FOR RANDOM FIELDS

(Presented by Academician A. N. Kolmogorov on 19 VI 1968)

1. Similarity considerations are especially useful for problems with strong interaction, when it is difficult to apply analytical techniques of the type of perturbation theory or the self-consistent-field method. If one regards as given a set of determining parameters on which the statistical characteristics of the field under consideration depend, then important results can be obtained by using only dimensional relations. A classical result of this kind is Kolmogorov–Obukhov' s "2/3 law" ([1,2]) for the structure of a turbulent velocity field. However, it is not always possible to guess in advance the determining parameters, which moreover may be different for different statistical characteristics. At the same time, if there is a uniform interaction mechanism in some range of scales, it is natural to suppose that the statistical characteristics of the field possess a certain similarity when passing from some scales to others. Such similarity, in contrast to similarity with respect to parameters, will be called scale similarity.

In the present article we shall restrict ourselves to considering scale similarity for nonnegative random fields. This problem arose in the study of the field of dissipation of kinetic energy in a turbulent flow. In ([3,4]) a logarithmically normal distribution law was proposed for the dissipation; moreover, in ([4]) a linear dependence was assumed of the variance of the logarithm of the averaged dissipation on the logarithm of the averaging scale. We note that the second part of paper ([4]) contains certain similarity hypotheses expressed in terms of the relation of differences of the turbulent velocity field. In ([5]) fragmentation coefficients were introduced (the ratio of the values of the field averaged over different scales) and similarity hypotheses were formulated for these coefficients. Further calculations in ([5]), as well as in ([6,7]), were carried out on the basis of a definite statistical model ("pulses within pulses"). In particular, it was shown that the spectral function of the dissipation field in a certain range of wave numbers has a power-law character with an exponent lying between 0 and -1. This result is in agreement with experimental data ([8–10]). In ([11]) the assumption was made that the logarithm of the fragmentation coefficient has a finite mean value and a finite variance. On this basis, by means of the central limit theorem, a logarithmically normal distribution law was obtained for the dissipation. Experimental data for quantities of the type of energy dissipation (quadratic in the gradients of velocity and temperature) give distributions close to the logarithmically normal one ([12–14]). We note that, within the framework of the model ([7]), in particular, the conditions necessary for the logarithmically normal law can be realized (when the pulses and the distribution density of the positions of subpulses have sufficiently long tails).

The works ([5–7,11]) are based on discrete fragmentation, when at each stage the scale changes by a definite number of times. This number n is in fact an additional parameter of the problem. In passing from the fragmentation number to a continuously varying wave number or distance, a certain mathematically nonrigorous trick is performed. A detailed calculation,

carried out in (7) shows that, under discrete subdivision in the spectrum, along with the power-law dependence on the wave number there appears a periodic dependence on the logarithm of the wave number, with a period equal to $\ln n$. An analogous circumstance also arises in the derivation of the logarithmically normal distribution. We note also that the restriction on the exponent in the correlation function* has so far been obtained only for a definite statistical model (5-7).

In the present article the problem is introduced in continuous form. It is shown that all moments of the field averaged over a certain scale have a power-law character. A restriction on the exponent of the correlation function is established independently of the particular model. A general expression is obtained for the characteristic function of the logarithm of the subdivision coefficient. The logarithmically normal distribution is, in this case, a definite asymptotic.

All the discussion is applicable not only to the field of dissipation of kinetic energy, but to any nonnegative field possessing similarity. In the case of the theory of turbulence these may be, for example, the square of the velocity curl or the square of the temperature gradient.

2. In an experimental study one usually deals with one-dimensional characteristics of a random field.** Two-dimensional or three-dimensional characteristics may be obtained from one-dimensional ones with allowance for the condition of statistical isotropy.

Consider a nonnegative random function $y(x)$ which, on scales smaller than some external scale L, possesses statistical homogeneity and isotropy (the two directions on the x axis are equivalent). Let us distinguish three mutually nested intervals with lengths $r < \rho < l < L$. Introduce the ratio of the values of the function $y(x)$ averaged over these intervals (the subdivision coefficient), for example,

$$q_{r,\rho} = y_r / y_\rho \qquad (q_{r,r} = 1). \tag{1}$$

For some interval $l_* < r < l < L$ (l_* is the inner scale) we shall require the following conditions to be satisfied: the subdivision coefficients $q_{r,\rho}$ and $q_{\rho,l}$ are statistically independent, and the probability distribution for each of these coefficients depends only on the ratio of the corresponding scales. These conditions define scale similarity (s.s.), and the corresponding interval is the s.s. interval.

It follows from the s.s. conditions that all moments

$$a_p(l/r) = \langle q_{r,l}^p \rangle \qquad (a_p(1) = 1) \tag{2}$$

(if they exist) must have a power-law character. Indeed, taking into account the identity

$$q_{r,l} = q_{r,\rho} q_{\rho,l} \tag{3}$$

we have

$$a_p(l/r) = a_p(\rho/r) a_p(l/\rho). \tag{4}$$

Hence, by virtue of the arbitrariness of ρ,

$$a_p(l/r) = (l/r)^{\mu_p}. \tag{5}$$

---

* If the correlation function in the range of scale similarity is proportional to $r^{-\mu}$ ($\mu > 0$), then, generally speaking, only for $\mu < 1$ will the spectrum be proportional to $k^{\mu-1}$ (k is the wave number). For $1 \leq \mu < 3$ the spectrum will have the indicated form only in the special case when the integral of the correlation function from 0 to ∞ (including the region of small distances where scale similarity is violated) is equal to zero. If, however, this integral is finite, then for $\mu \geq 1$ the spectrum is constant in the interval of scale similarity ("white noise").

** In studying a turbulent flow one obtains a record of the values of some quantity, measured at a fixed point, as a function of time. Then, using the "frozen" hypothesis, this record is interpreted as the distribution of the given quantity along a line parallel to the mean velocity of the flow and passing through the observation point.

From the obvious inequality $q_{r,l} \leq l/r$ it follows that $\mu_p \leq p$. Consider the quantities

$$q_i = n \frac{\displaystyle\int_{(i-1)r}^{ir} y(x)\,dx}{\displaystyle\int_0^{nr} y(x)\,dx} \quad (i = 1, \dots, n), \tag{6}$$

which satisfy the condition

$$\sum_{i=1}^{n} q_i = n. \tag{7}$$

Averaging (7), we obtain

$$a_1(n) = \langle q_i \rangle = 1, \qquad \mu_1 = 0, \qquad \langle q_{r,l} \rangle = 1. \tag{8}$$

Now square both sides of (7) and average. We have:

$$a_2(n) = n - \frac{1}{n} \sum_{i,j=1,\,(i \neq j)}^{n} \langle q_i q_j \rangle. \tag{9}$$

Since the quantities q_i are nonnegative, the sum appearing on the right-hand side of (9) is also nonnegative and can be equal to zero only in the case when all mutual moments between different a_i vanish simultaneously. In fact, this would mean that the function $y(x)$ is a δ-function. Excluding this case from consideration, we have

$$a_2(n) < n, \qquad \mu_2 < 1. \tag{10}$$

On the other hand, in the presence of fluctuations, evidently, $\mu_2 > 0$. Thus:

$$\langle y_r^2 / y_l^2 \rangle = (l/r)^{\mu} \quad (\mu \equiv \mu_2), \qquad 0 < \mu < 1. \tag{11}$$

It is natural to put $y_L \approx \langle y \rangle$; then (11) gives

$$\langle y_r^2 \rangle = \langle y \rangle^2 (L/r)^{\mu}, \tag{12}$$

however, in this formula there may already appear a factor of order unity depending on the large-scale features of the field. Using the simple relation indicated in (6),

$$\langle y(x+r) y(x) \rangle = \frac{1}{2} \frac{d^2}{dr^2} \left[r^2 \langle y_r^2 \rangle \right], \tag{13}$$

we obtain

$$\langle y(x+r)y(x)\rangle = \tfrac{1}{2}(2-\mu)(1-\mu)\langle y_r^2\rangle \sim r^{-\mu}. \tag{14}$$

For $r \ll L$, evidently, $\langle y_r^2\rangle \gg \langle y\rangle^2$; therefore

$$b(r) = \langle y'(x+r)y'(x)\rangle \sim r^{-\mu} \qquad (y' = y - \langle y\rangle). \tag{15}$$

For the spectrum, taking into account the restriction (11), we obtain

$$\varphi(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikr} b(r)\,dr \sim k^{\mu-1} \qquad (0<\mu<1), \tag{16}$$

where k is the wave number. Experimental data ([8–10]) give the value $\mu \approx 0.4$.

3. Let $W(q, l/r)$ denote the distribution density of the quantity $q_{r,l}$. From the scale-similarity conditions, taking account of identity (3), it is not difficult to obtain the equation

$$W\left(q, \frac{l}{r}\right) = \int W\left(\frac{q}{\eta}, \frac{\rho}{r}\right) W\left(\eta, \frac{l}{\rho}\right)\frac{d\eta}{\eta}. \tag{17}$$

It is convenient to introduce the characteristic function for the logarithm of the fragmentation coefficient

$$\psi(s, l/r) = \langle \exp\{isz_{r,l}\}\rangle, \qquad z_{r,l} = \ln q_{r,l}. \tag{18}$$

The requirements of m.s. taking into account the identity

$$z_{r,l} = z_{r,\rho} + z_{\rho,l} \qquad (r \le \rho \le l) \tag{19}$$

give

$$\psi(s, l/r) = \psi(s, \rho/r)\psi(s, l/\rho). \tag{20}$$

By virtue of the arbitrariness of ρ, we obtain

$$\psi(s, l/r) = (l/r)^{-\alpha(s)} = \exp\{-\alpha(s)\ln l/r\}. \tag{21}$$

The function $\alpha(s)$ entering into (21) satisfies the conditions

$$\alpha(0) = 0, \qquad \alpha(-i) = 0, \qquad \alpha(-2i) = -\mu. \tag{22}$$

The first condition is the normalization of the total probability, the second follows from (8), and the third from (11).

If the function $\alpha(s)$ has a certain number of derivatives at zero, $\alpha^{(p)}(0)$, then there exists the corresponding number of cumulants of the distribution of the quantity $z_{r,l}$, and all of them are proportional to the logarithm of the ratio of scales:

$$\chi_p\left(\frac{l}{r}\right) = (i)^{-p}\, \frac{d^p \ln\psi(s, l/r)}{ds^p}\bigg|_{s=0} = (i)^{2-p}\alpha^{(p)}(0)\ln\frac{l}{r}. \tag{23}$$

Let us introduce the quantities

$$\zeta_{r,l} = \frac{z_{r,l} - \chi_1(l/r)}{\chi_2^{1/2}(l/r)}, \qquad \chi\left(t, \frac{l}{r}\right) = \langle \exp\{it\zeta_{r,l}\}\rangle. \tag{24}$$

It is not difficult to show that, as $\ln(l/r) \to \infty$,

$$\chi(t, l/r) \to \exp\{-{}^1/2t^2\} \tag{25}$$

uniformly on any finite interval $|t| \leq T$. Thus, asymptotically as $\ln(l/r) \to \infty$, the fragmentation coefficient $q_{r,l}$ has a logarithmically normal distribution; moreover, from (22) it follows that

$$\alpha^{(1)}(0) = {}^1/2i\mu, \qquad \alpha^{(2)}(0) = \mu, \qquad 0 < \mu < 1. \tag{26}$$

The cumulants of the logarithm of the fragmentation coefficient may fail to exist, for example, if the field at certain scales may turn to zero with finite probability [5–7]. In this case, and also when the logarithm of the ratio of scales is not very large, one must use the more general expression (21).

Institute of Atmospheric Physics
Academy of Sciences of the USSR

Received
14 VI 1968

References

[1] A. N. Kolmogorov, DAN, 30, No. 4 (1941).
[2] A. M. Obukhov, DAN, 32, No. 1 (1941).
[3] A. M. Obukhov, J. Fluid Mech., 13, No. 1 (1962).
[4] A. N. Kolmogorov, J. Fluid Mech., 13, No. 1 (1962).
[5] E. A. Novikov, R. W. Stewart, Izv. AN SSSR, Ser. Geophys., No. 3 (1964).
[6] E. A. Novikov, Phys. Atmos. and Ocean, 1, No. 8 (1965).

[7] E. A. Novikov, DAN, 168, No. 6 (1966).
[8] A. S. Gurvich, S. L. Zubkovskii, Izv. AN SSSR, Ser. Geophys., No. 12 (1963).
[9] A. S. Gurvich, S. L. Zubkovskii, Phys. Atmos. and Ocean, 1, No. 8 (1965).
[10] S. Pond, R. W. Stewart, Phys. Atmos. and Ocean, 1, No. 9 (1965).
[11] A. M. Yaglom, DAN, 166, No. 1 (1966).
[12] A. S. Gurvich, Phys. Atmos. and Ocean, 2, No. 10 (1966).
[13] A. S. Gurvich, DAN, 172, No. 3 (1967).
[14] A. S. Gurvich, A. M. Yaglom, Phys. Fl., 10, Suppl. (1967).

Note: Figure translations are in progress. See original paper for figures.