TORSION OF AN ELASTIC LAYER BY TWO PUNCHES

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Abstract

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THEORY OF ELASTICITY

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TORSION OF AN ELASTIC LAYER BY TWO PUNCHES

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In the present note a mixed boundary-value problem of the theory of elasticity is considered for an infinite layer twisted by two coaxial punches of different radii.

If it is assumed that, in cylindrical coordinates, the points of the layer belong to the region $0 \le r < \infty$, $-h \le z \le 0$, then the problem posed reduces to determining the single nonzero component of the displacement vector $u_\varphi(r,z) = u$, satisfying the equation

$$\Delta u - u/r^2 = 0 \tag{1}$$

and the boundary conditions

$$u\big|_{r<a} = \vartheta_a r, \qquad \partial u/\partial z\big|_{r>a} = 0, \qquad z = 0,$$

$$u\big|_{r<b} = -\vartheta_b r, \qquad \partial u/\partial z\big|_{r>b} = 0, \qquad z = -h. \tag{2}$$

Here a and b are the radii, and ϑ_a and ϑ_b are the angles of rotation of the punches. It is also assumed that $\lim_{r\to\infty} u = 0$.

We shall seek the solution of the problem in the form of a Hankel integral

$$u = \int_0^\infty [C(\lambda)\operatorname{sh}\lambda z + D(\lambda)\operatorname{sh}\lambda(h+z)]J_1(\lambda r)\frac{d\lambda}{\operatorname{sh}\lambda h}. \tag{3}$$

In this case the boundary conditions lead to the following system of paired integral equations:

$$\int_0^\infty DJ_1(\lambda r)\,d\lambda = \vartheta_a r, \qquad 0 \le r < a; \tag{4}$$

$$\int_0^\infty \lambda(C + D \operatorname{ch} \lambda h) J_1(\lambda r) \frac{d\lambda}{\operatorname{sh} \lambda h} = 0, \qquad a < r < \infty; \tag{5}$$

$$\int_0^\infty C J_1(\lambda r)\, d\lambda = -\vartheta_b r, \qquad 0 \le r < b; \tag{6}$$

$$\int_0^\infty \lambda(C \operatorname{ch} \lambda h + D) J_1(\lambda r) \frac{d\lambda}{\operatorname{sh} \lambda h} = 0, \qquad b < r < \infty. \tag{7}$$

The homogeneous equations (5), (7) can be satisfied by setting (^{1})

$$C + D \operatorname{ch} \lambda h = \operatorname{sh} \lambda h \int_0^a \varphi(t) \sin \lambda t \, dt,$$

$$C \operatorname{ch} \lambda h + D = \operatorname{sh} \lambda h \int_0^b \omega(t) \sin \lambda t \, dt, \tag{8}$$

where φ and ω are new unknown functions, which are assumed to be continuously differentiable.

Finding from (8) the values of $C(\lambda)$ and $D(\lambda)$, substituting them into (4), (6), and using the relations (2)

$$\int_0^\infty J_1(\lambda r) \sin \lambda t \, d\lambda = \frac{t}{r\sqrt{r^2 - t^2}}, \qquad r > t; \tag{9}$$

$$J_1(z) = \frac{2}{\pi} \int_0^{\pi/2} \sin(z \sin\theta) \sin\theta \, d\theta, \tag{10}$$

we arrive at the relations

$$\int_0^{\pi/2} r \sin\theta \left[\varphi(r \sin\theta) + \frac{2}{\pi} \int_0^a \varphi(\tau) \Phi_1(r\sin\theta, \tau)\, d\tau - \frac{2}{\pi} \int_0^b \omega(\tau) \Phi_2(r \sin\theta, \tau)\, d\tau \right] d\theta = \vartheta_a r^2, \qquad r < a; \tag{11}$$

$$\int_0^{\pi/2} r \sin\theta \left[\omega(r \sin\theta) + \frac{2}{\pi} \int_0^a \omega(\tau) \Phi_1(r\sin\theta, \tau)\, d\tau - \frac{2}{\pi} \int_0^b \varphi(\tau) \Phi_2(r \sin\theta, \tau)\, d\tau \right] d\theta = -\vartheta_b r^2, \qquad r < b, \tag{12}$$

where it is set that

$$\Phi_1(t,\tau) = \int_0^\infty e^{-\lambda h} \sin\lambda t \sin\lambda\tau \, \frac{d\lambda}{\operatorname{sh}\lambda h}, \qquad \Phi_2(t,\tau) = \int_0^\infty \frac{\sin\lambda t \sin\lambda\tau}{\operatorname{sh}\lambda h}\, d\lambda. \tag{13}$$

Solving the Schläfli equations (³) (11), (12), we obtain the system of integral equations

$$\varphi(t) + \frac{2}{\pi}\int_0^a \varphi(\tau)\Phi_1(t,\tau)\,d\tau - \frac{2}{\pi}\int_0^b \omega(\tau)\Phi_2(t,\tau)\,d\tau = \frac{4}{\pi}\vartheta_a t, \qquad 0 < t < a,$$

$$\omega(t) + \frac{2}{\pi}\int_0^b \omega(\tau)\Phi_1(t,\tau)\,d\tau - \frac{2}{\pi}\int_0^a \varphi(\tau)\Phi_2(t,\tau)\,d\tau = -\frac{4}{\pi}\vartheta_b t, \qquad 0 < t < b. \tag{14}$$

Thus, the posed problem has been reduced to a system of regular Fredholm integral equations. It is essential to note that the symmetric kernels (13) of the indicated system (14), with the aid of the formula (4)

$$\int_0^\infty e^{-\beta x}\frac{\sin\alpha x}{\operatorname{sh}\gamma x}\,dx = \frac{1}{2i\gamma}\left[\psi\!\left(\frac{\beta+\gamma+i\alpha}{2\gamma}\right) - \psi\!\left(\frac{\beta+\gamma-i\alpha}{2\gamma}\right)\right] \tag{15}$$

are expressed explicitly in terms of the psi function

$$\begin{aligned} \Phi_1(t,\tau) &= \frac{1}{2h}\operatorname{Re}\left\{\psi\left[1+\frac{(t+\tau)i}{2h}\right] - \psi\left[1+\frac{(t-\tau)i}{2h}\right]\right\}, \\ \Phi_2(t,\tau) &= \frac{1}{2h}\operatorname{Re}\left\{\psi\left[\frac{1}{2}+\frac{(t+\tau)i}{2h}\right] - \psi\left[\frac{1}{2}+\frac{(t-\tau)i}{2h}\right]\right\}. \end{aligned} \tag{16}$$

In order to express the angles of rotation ϑ_a and ϑ_b that enter the solution in terms of the twisting moments M applied to the punches, by forming the expres-

···for the tangential stresses $\tau_{\varphi z}$ acting under the punches. Since $\tau_{\varphi z} = G\,\partial u/\partial z$, where G is the shear modulus, we have

$$\frac{1}{G}\tau_{\varphi z}\Big|_{\substack{z=0\\ r<a}} = \int_0^\infty \lambda(C + D\operatorname{ch}\lambda h)J_1(\lambda r)\frac{d\lambda}{\operatorname{sh}\lambda h} =$$

$$= \int_0^\infty \lambda J_1(\lambda r)\,d\lambda \int_0^a \varphi(t)\sin\lambda t\,dt = -\frac{d}{dr}\int_r^a \frac{\varphi(t)\,dt}{\sqrt{t^2-r^2}}.$$

The condition of statics gives

$$M = -2\pi G \int_0^a r^2 \, d\left[\int_r^a \frac{\varphi(t)\, dt}{\sqrt{t^2 - r^2}}\right] = 4\pi G \int_0^a r\, dr \int_r^a \frac{\varphi(t)\, dt}{\sqrt{t^2 - r^2}} = 4\pi G \int_0^a t\varphi(t)\, dt.$$

Carrying out analogous calculations for the punch of radius b, we finally find

$$M = 4\pi G \int_0^a \varphi(t) t\, dt = 4\pi G \int_0^b \omega(t) t\, dt, \tag{17}$$

which is the system of equations for the desired angles ϑ_a and ϑ_b.

Let us note in conclusion that in the particular case $a = b$, corresponding to the torsion of a layer of thickness $h/2$ with fixed base $z = -h/2$, system (14) reduces to one integral equation for the function $\varphi(t) = -\omega(t)$, obtained by us earlier in work [1].

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Note: Figure translations are in progress. See original paper for figures.

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