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1969

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Abstract

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P. FATOU' S THEOREM FOR REGULAR QUATERNION FUNCTIONS

(Presented by Academician I. N. Vekua, 14 VIII 1968)

The well-known theorem of P. Fatou consists in the assertion that a function $F(z)$, analytic and bounded inside the circle $|z| < 1$, has, at almost every point of the circle $|t| = 1$, a definite limit $F(t)$ when the point z approaches the point t along any nontangential path; $F(z)$ is determined (for $|z| < 1$) in the form of the Cauchy integral of its boundary values $F(t)$.

In the present work an analogue of this Fatou theorem is constructed for the case of regular quaternion functions $f(z)$, i.e., for continuously differentiable functions of four real variables, defined in a certain domain G of four-dimensional Euclidean space E_4 with values in the algebra of quaternions, satisfying in G the relations

$$fD \equiv 0 \tag{1}$$

or

$$Df \equiv 0, \tag{2}$$

where D is an operator of the form

$$D = \partial/\partial x_0 + i_1\partial/\partial x_1 + i_2\partial/\partial x_2 + i_3\partial/\partial x_3.$$

Quaternion functions satisfying relation (1) are called right-regular, those satisfying relation (2)—left-regular. All the results given below are formulated for right-regular functions, although analogous results are also valid for left-regular functions.

The main result of the work may be formulated as follows:

Theorem 1. *Let the quaternion function $f(z)$ be right-regular in a star-shaped domain G, bounded by a Lyapunov hypersurface S, and let it satisfy there the condition*

$$\sup_{0<r_n<1} \iiint_{S_{r_n}} |f(t)|\, dS_{r_n} \leq M,$$

where M is a finite constant; S_{r_n} is a family of hypersurfaces homothetic to S with homothety coefficient r_n and with some center of homothety z^*, satisfying the condition: every ray issuing from the point z^* intersects the hypersurface S at only one point.

Then, for almost every point of the hypersurface S, the function $f(z)$ has angular limiting values $f(t)$, and for any point $z \in G$, $f(z)$ can be represented in the form

$$f(z) = \frac{1}{8\pi^2} \iiint_S f(t)\, dZ\, \Delta_t[(t-z)^{-1}], \tag{3}$$

where

$$dZ = (\cos(\widehat{n, Ox_0}) + i_1 \cos(\widehat{n, Ox_1}) + i_2 \cos(\widehat{n, Ox_2}) + i_3 \cos(\widehat{n, Ox_3}))\, dS,$$

with n the vector of the inner normal to S at the point t; dS the element of volume of the hypersurface S; Δ_t the Laplace operator.

For the proof of this theorem we shall need one auxiliary result, for the formulation of which we introduce some preliminary concepts.

1) Let S be a closed Lyapunov hypersurface in E_4. On S there is given a completely additive class A of sets e, on which a quaternion-valued countably additive set function $\Phi(e)$ (a Φ-measure) of bounded variation on S is defined. In addition, we assume that the sets $e \in A$ are Lebesgue measurable. Then the expression

$$\frac{1}{8\pi^2} \iiint_S d\Phi \Delta_t[(t-z)^{-1}],$$

where z is an arbitrary point of E_4, $t \in S$, will be called the **right quaternion Cauchy-Stieltjes type integral**.

2) The **right generalized derivative of the function** $\Phi(e)$ **at a point** $t \in S$ **with respect to the class** A **of sets** e will be called the expression

$$D\Phi(t) = D_A\Phi(t) = \lim_{n\to\infty} \Phi(e_n) \left(\iiint_{e_n} dZ \right)^{-1},$$

where e_n $(n = 0, 1, 2, \ldots)$ is any sequence from A which contracts regularly to the point t.

Now let the points z_1 and z_2 lie on one straight line passing through the point $t' \in S$ and inclined to the normal $n(t')$ at an angle ψ, with z_1 inside S, z_2 outside S, and at distance ε from t'. Then the following is true.

Lemma. *The difference of the values of the right quaternion Cauchy–Stieltjes type integral inside and outside S*

$$\frac{1}{8\pi^2}\iiint_S d\Phi\Delta_t\big[(t-z_1)^{-1}\big]-\frac{1}{8\pi^2}\iiint_S d\Phi\Delta_t\big[(t-z_2)^{-1}\big]$$

tends to the limit $D\Phi(t')$, as ε tends to zero, for all points t' of the hypersurface S, except, possibly, for points of a set of measure zero, independent of ψ, uniformly with respect to ψ, $|\psi|\le\frac{\pi}{2}m,\ m<1$.

We omit the proof of this lemma. It can be obtained by following the scheme of the proof of the corresponding result for the case of analytic functions of one complex variable, proposed by I. I. Privalov in paper (^{1}).

Proof of Theorem 1. Without loss of generality we may assume that the domain G contains the origin and that the point $z^*=0$ is the center of homothety specified in the hypothesis of the theorem.

Consider the function $F_n(z)=f(r_nz)$, $0<r_n<1$. Since, by the hypothesis of the theorem, the function $f(z)$ is right-regular in the domain G bounded by the hypersurface S, the function $F_n(z)$ is right-regular in a domain wider than G, and hence also on the hypersurface S.

The values of the function $F_n(z)$ at the points $z\in S$ obviously coincide with the values of the function $f(z)$ at the points $z'=r_nz$, situated on the hypersurface S_{r_n}, lying inside S and homothetic to it with homothety coefficient r_n and center of homothety $z^*=0$.

But then, from the hypothesis of the theorem and from the fact that the values dZ on S and on S_{r_n} differ only by the volume elements dS and dS_{r_n}, which are obviously related by

$$dS=\frac{1}{r_n^3}\,dS_{r_n},$$

it follows that the integrals

$$\iiint_e F_n(s)\,dZ \tag{4}$$

exist and

$$\iiint\limits_e |F_n(t)\,dZ|\leqslant M.$$

Here the sets e belong to the class A defined above.

The integral (4) assigns to each set $e \in A$ a quaternion, and we may assume that a quaternionic measure μ is given on S. Obviously, to each r_n there corresponds a measure taken in this way, and therefore we have a family of measures $\{\mu_n\}$.

To each quaternionic measure μ_n from this family there corresponds a right-homogeneous additive continuous functional (2) in the space of functions $\{\varphi(t)\}$ continuous on S,

$$\iiint\limits_S F_n(t)\, dZ \varphi(t). \tag{5}$$

Since the norm of these functionals is bounded by one and the same constant, in view of what was said earlier, it follows from (3) that the family of right-homogeneous additive continuous functionals (5) is weakly compact in the space of functions $\{\varphi(t)\}$ continuous on S.

Take an arbitrary weakly convergent sequence of functionals from the family (5). It converges weakly to some right-homogeneous additive continuous functional. To this limiting functional there corresponds a certain quaternionic measure μ (of bounded variation on S).

Let us now consider the right quaternionic Cauchy–Stieltjes integral

$$\frac{1}{8\pi^2} \iiint\limits_S d\mu\, \Delta_t[(t-z)^{-1}]. \tag{6}$$

This integral defines, inside S, a certain right-regular function. We shall show that this right-regular function is $f(z)$.

Let S' be a hypersurface homothetic to S with homothety coefficient $r_n < 1$ and center of homothety $z^* = 0$. Obviously, S' lies inside S. To each point $t \in S$ there corresponds the point $t' = r_{nt}$. The function $f(z)$ is right-regular inside S' and on S. Therefore, by the second fundamental theorem (4) we have:

$$f(z) = \frac{1}{8\pi^2} \iiint\limits_{S'} f(t')\, dZ \Delta_{t'}[(t'-z)^{-1}] = \frac{1}{8\pi^2} r_n^3 \iiint\limits_S F_n(t)\, dZ \Delta_{r_{nt}}[(r_{nt}-z)^{-1}].$$

It is not difficult to show that for each fixed point $z \in G$ and an arbitrarily small $\varepsilon > 0$ the relation

$$\left| \frac{1}{8\pi^2} \iiint\limits_S d\mu\, \Delta_t[(t-z)^{-1}] - f(z) \right| < \varepsilon. \tag{7}$$

holds. Indeed:

$$\left|\frac{1}{8\pi^2}\iiint\limits_S d\mu\,\Delta_t[(t-z)^{-1}]-f(z)\right|\leqslant$$

$$\leqslant\frac{1}{8\pi^2}\left|\iiint\limits_S d\mu\,\Delta_t[(t-z)^{-1}]-r_n^3\iiint\limits_S d\mu\,\Delta_t[(t-z)^{-1}]\right|+$$

$$+\frac{1}{8\pi^2}\left|r_n^3\iiint\limits_S d\mu\,\Delta_t[(t-z)^{-1}]-r_n^3\iiint\limits_S F_n(t)\,dZ\Delta_t[(t-z)^{-1}]\right|+$$

$$+\frac{1}{8\pi^2}\left|r_n^3\iiint\limits_S F_n(t)\,dZ\Delta_t[(t-z)^{-1}]-r_n^3\iiint\limits_S F_n(t)\,dZ\Delta_{r_{nt}}[(r_{nt}-z)^{-1}]\right|=A+B+C.$$

Obviously, A, B, and C, for r_n sufficiently close to 1, are arbitrarily small, whence inequality (7) follows. Thus, at every point $z\in G$ we have

$$f(z)=\frac{1}{8\pi^2}\iiint\limits_S d\mu\,\Delta_t[(t-z)^{-1}]. \tag{8}$$

In exactly the same way one can show that, at any point z external to S, the integral (6) is equal to zero.

Applying the lemma to the integral (6) and using the fact of the existence almost everywhere on S of the right generalized derivative $D\mu(t)$ relative to the class A of sets—which, in turn, follows from (5)—we are convinced that, under approach along arbitrary nontangential paths in G to boundary points of S, $f(z)$ almost everywhere on S assumes definite boundary values $f(t)=D\mu(t)$.

But then

$$\mu=\iiint f(t)\,dZ=0,$$

and from (8) equality (3) follows. The theorem is completely proved.

Remark. The lemma and Theorem 1 are, obviously, also valid in the case when the hypersurface S consists of a finite number of pieces of Lyapunov hypersurfaces.

Theorem 1 is also valid in the case when the domain G can be divided into a finite number of star-shaped domains, each of which is bounded by a hypersurface composed of pieces of Lyapunov hypersurfaces.

Theorem 1 makes it possible to prove a number of facts about limiting angular values of right-regular functions. We present some of them.

For simplicity, in what follows we shall speak only of star-shaped domains bounded by a Lyapunov hypersurface.

Theorem 2. *If a right-regular function $f(z)$ is bounded inside the domain G, then*

$$\iiint\limits_S f(t)\,dz = 0,$$

where $f(t)$ are the limiting angular values of $f(z)$ on S.

The analogue of natural powers z in the theory of regular quaternionic functions is given by polynomials of the form

$$p_{n_1n_2n_3}(z) = \frac{1}{n!}\sum_{k_r}(x_{k_1} - i_{k_1}x_0)\cdots(x_{k_n} - i_{k_n}x_0),$$

where $n_1 + n_2 + n_3 = n$; n_1 of the numbers $k_1, k_2, \dots, k_n$ are equal to 1; n_2 of these numbers are equal to 2, and n_3 of these numbers are equal to 3; the summation is over the distinct permutations of the numbers $k_1, k_2, \dots, k_n$.

Theorem 3. *Let an integrable quaternionic function $f(z)$ be given on a closed hypersurface S; if*

$$\iiint\limits_S f(t)\,dZ\,p_{n_1n_2n_3}(t) = 0$$

for $n = 0, 1, 2, \ldots$, then $f(t)$ represents on S the angular limiting values of some function right-regular inside S.

Let us note in conclusion that Theorem 1 can be applied to the study of certain types of singularities of regular quaternionic functions.

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Received
22 VII 1968

CITED LITERATURE

1. I. I. Privalov, *Boundary Properties of Analytic Functions*, Moscow-Leningrad, 1950.

2. R. Kh. Kristalinskii, "Theory of functions, functional analysis and their applications," **4**, 46 (1967).

3. L. P. Kantorovich, G. P. Akilov, *Functional Analysis in Normed Spaces*, Moscow, 1959.

4. R. Fueter, *Comm. Math. Helv.*, **7**, 307 (1934-1935).

5. G. E. Shilov, B. A. Gurevich, *Integral, Measure, Derivative*, Moscow, 1967.

Note: Figure translations are in progress. See original paper for figures.

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