MATHEMATICS

S. G. SAMKO

1969

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. G. SAMKO

ON SOLVABILITY IN CLOSED FORM OF SINGULAR INTEGRAL EQUATIONS

(Presented by Academician A. A. Dorodnitsyn, 24 IV 1969)

Relatively few cases are known of solutions in closed form of singular integral equations:

$$a(t)\varphi(t) + \frac{1}{\pi i}\int_\Gamma \frac{M(t,\tau)}{\tau - t}\varphi(\tau)\,d\tau = f(t). \tag{1}$$

These are, first of all, the case of the characteristic equation and the equation conjugate to it (1,2), as well as equations with an automorphic kernel (2–3). A number of authors (6–8) have solved certain equations of the form (1) in closed form in other cases as well. The arguments in these works were based on analytic properties of the functions $M(t,\tau)$, $a(t)$.

In the present note the involutivity of the singular integral operator is proved in the case where the kernel is analytic, and a new general class of equations of the form (1), solvable in closed form, is investigated. This class contains, in particular, the equations considered in (6–8), and also, for example, equations obtained from the characteristic equation $a\varphi + bS\varphi = f$ by perturbing it with an operator K whose kernel is analytically continuable up to the factor $[a(t) + b(t)]^{-1}$, and other equations.

§ 1. Let Γ be a simple closed contour* bounding a simply connected domain D^+. Everywhere below it is assumed that:

1) $M(t,\tau) \in C(\Gamma \times \Gamma)$;

2) $|M(t,\tau) - M(t,t)| < A|t-\tau|^\varepsilon,\ \varepsilon > 0,\ t,\tau \in \Gamma.$

Introduce the notation:

$$\mathcal{K}(t,\tau) = \frac{M(t,\tau) - M(t,t)}{\tau - t}, \qquad K\varphi = \frac{1}{\pi i}\int_\Gamma \mathcal{K}(t,\tau)\varphi(\tau)\,d\tau, \tag{2}$$

$$S\varphi = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)\,d\tau}{\tau - t}, \qquad S_M\varphi = \frac{1}{\pi i}\int_\Gamma \frac{M(t,\tau)}{\tau - t}\varphi(\tau)\,d\tau. \tag{3}$$

In what follows $\varphi \in B$, where B denotes either the space $\mathcal{L}_p(\Gamma)$, $p > 1$, or the space of Hölder functions $H^\lambda(\Gamma)$, $0 < \lambda < 1$.

Lemma. *If the function $M(t,\tau)$ is analytically continuable into the domain D^+ in both variables and $M(t,t) \equiv 1$, then the operator S_M is involutive:*

$$S_M^2 = I.$$

The proof of the lemma is obtained by decomposing B into the direct sum $B = B^+ \oplus B^-$ by the projectors ${}^1/_2(\pm I + S)$. From the conditions imposed on $M(t,\tau)$ it follows that $K\varphi \in B^+$ for $\varphi \in B$ and that $K\varphi^+ = 0$ for $\varphi^+ \in B^+$. Therefore $(I - S)K = 0$ and $K(I + S) = 0$. Consequently,

$$KS = -K, \qquad SK = K,$$

* Piecewise smooth if Hölder functions on Γ are considered, and piecewise Lyapunov if functions from $\mathcal{L}_p(\Gamma)$ are considered.

and then $(KS)(SK) = K^2$. Since $S^2 = I$, it follows that $K^2 = 0$. Further, since $M(t,t) = 1$, we have $S_M = K + S$, and then $S_M^2 = S^2 + SK + KS = I$.

Corollary. If the function $M(t,\tau)/M(t,t)$ is analytically continuable in both variables into the domain D^+, then the equation $S_M\varphi = f$ is solvable in closed form.

2. Let us turn to the equation of the general form (1). We denote, as usual, $M(t,t) = b(t)$ and shall start from the equation in the form

$$N\varphi = a\varphi + bS\varphi + K\varphi = f. \tag{1'}$$

We shall assume that

$$a, b \in \begin{cases} C,\ B = \mathcal{L}_p, \\ H^\lambda,\ B = H^\lambda; \end{cases} \qquad a(t) \pm b(t) \neq 0, \quad t \in \Gamma,$$

and that K is an operator of the form (2) with kernel $\mathcal{K}(t,\tau)$, having on Γ a weak singularity:

$$|\mathcal{K}(t,\tau)| < A|t-\tau|^{\varepsilon-1}.$$

By $\varkappa$ we denote the Cauchy index of the function

$$G(t) = [a(t) - b(t)]/[a(t) + b(t)],$$

and by α the nullity of the operator N in the space B.

We shall show that equation $(1')$ is solvable in closed form in the following cases:

I. The function $\mathcal{K}(t,\tau)/(a(t)+b(t))$ is analytic in τ and meromorphic in t in the domain D^+.

II. The function $b(t)/a(t)$ is meromorphic in D^+, and one of the functions $\mathcal{K}(t,\tau)/a(t)$, $\mathcal{K}(t,\tau)/b(t)$ is analytic in τ and meromorphic in t in D^+.

Taking into account that $K\varphi^+ = 0$ in each of the cases I, II $\left(\varphi^{\pm} = \pm\frac{1}{2}\varphi + \frac{1}{2}S\varphi\right)$, we reduce equation $(1')$ to a relation of the type of the Riemann problem ([2]):

$$\varphi^+ + \frac{1}{a+b}K\varphi^- = G\varphi^- + g, \qquad g = \frac{f}{a+b}. \tag{4}$$

In case I we have the representation

$$\frac{\mathcal{K}(t,\tau)}{a(t)+b(t)} = \frac{A_1^+(t,\tau)}{\Pi_{a+b}^+(t)}, \qquad \Pi_{a+b}^+(t) = \prod_{k=1}^{n_1}(t-z_k)^{m_k}, \tag{5}$$

where $z_k \in D^+$, m_k are positive integers, and the function $A_1^+(t,\tau)$ is analytically continuable into D^+ with respect to t and τ. Relation (4) takes the form

$$\Pi_{a+b}^+(t)\varphi^+(t) + (A_1^+\varphi^-)(t) = \Pi_{a+b}^+(t)G(t)\varphi^-(t) + g(t), \tag{6}$$

where A_1^+ is the integral operator with kernel $A_1^+(t,\tau)$. Since $A_1^+\varphi^- \in B^+$, the last relation is an ordinary Riemann problem, from which the functions

$$\Pi_{a+b}^+\varphi^+ + A_1^+\varphi^-$$

and φ^-, and consequently also $\varphi(t)$, are determined in closed form. Factoring, as usual ([1,2]), the multiplier

$$G(t) = t^{\varkappa}\chi^+(t)/\chi^-(t)$$

and bringing (6) to a form allowing the application of Liouville’ s theorem, we arrive at a polynomial of degree

$$\varkappa - 1 + \sum_{k=1}^{n_1} m_k$$

with arbitrary coefficients (in the case $\varkappa + \sum m_k > 0$). However, the presence of the factor $\Pi_{a+b}^+(t)$ in front of $\varphi^+(t)$, which vanishes in D^+ with total order of zeros $\sum m_k$, leads to the fact that

$$\max(\varkappa, 0) \le \alpha \le \max\left(\varkappa + \sum_{k=1}^{n_1} m_k, 0\right).$$

In case II let us consider, for example, the variant when the function $\mathcal{K}(t,\tau)/a(t)$ is meromorphic. We write representations of type (4) for the functions $\mathcal{K}(t,\tau)/a(t)$ and $G(t)$ (the meromorphicity of $G(t)$ follows from the meromorphicity of $b(t)/a(t)$):

$$\frac{\mathcal{K}(t,\tau)}{a(t)} = \frac{A_2(t,\tau)}{\Pi_a^+(t)}, \qquad \Pi_a^+(t) = \prod_{k=1}^{n_2}(t-a_k)^{p_k}, \qquad a_k \in D^+,$$

$$G(t) = \frac{G^+(t)}{\Pi_G^+(t)}, \qquad \Pi_G^+(t) = \prod_{k=1}^{n_3}(t-g_k)^{r_k}, \qquad g_k \in D^+.$$

Relation (4) is reduced to the form

$$\Pi_a^+\Pi_G^+\varphi^+ + \frac{1}{2}\Pi_G^+A_2^+\varphi^- + \frac{1}{2}G^+A_2^+\varphi^- = G^+\Pi_a^+\varphi^- + g\Pi_a^+\Pi_G^+,$$

where A_2^+ is the operator with kernel $A_2^+(t,\tau)$. We have again arrived at a Riemann problem, from which the unknown function is determined in closed form.

As in case I, we obtain

$$\max(\varkappa, 0) \le \alpha \le \max\left(\varkappa + \sum_{k=1}^{n_2} p_k + \sum_{k=1}^{n_3} r_k,\ 0\right).$$

The case of meromorphy of the function $\mathcal{K}(t,\tau)/b(t)$ is considered analogously. Let us note that in case II the canonical function $\chi^-(t)$ is always rational in D^-.

3. From II let us single out the case when the function $\mathcal{K}(t,\tau)/b(t)$ is analytic in D^+ with respect to t and to τ, and the function $G(t)$ is also analytic in D^+; and suppose, for simplicity, that $G(t)$ vanishes only at one point of the domain D^+, for example at $t = 0 \in D^+$. Then $\chi^-(t) \equiv 1$, and for $\varkappa > 0$ the zeros of the operator N have the form

$$\varphi_j(t) = \frac{1}{a(t)+b(t)}\left\{\frac{b(t)}{t^{1+j}} - \frac{1}{j!}\left.\frac{\partial^j \mathcal{K}}{\partial \tau^j}\right|_{\tau=0}\right\}, \tag{7}$$

$$j = 0, 1, \dots, \varkappa - 1,$$

and for $\varkappa < 0$ the solvability conditions consist in the orthogonality of $f(\tau)$ to the functions

$$\psi_j(\tau) = \frac{1}{a(\tau) - b(\tau)} \left\{ \frac{1}{\tau^{1+j}} + \frac{1}{j!} \frac{\partial^j}{\partial \tau^j} \left(\frac{\mathcal{K}(t,\tau)}{b(t)} \right) \Bigg|_{t=0} \right\}, \tag{8}$$

$$j = 0, 1, \dots, |\varkappa| - 1.$$

The resolvent R of the operator $(1')$ has the form

$$Rf = \frac{a}{a^2 - b^2} f - \frac{1}{a-b}(bS + K)\frac{f}{a-b}.$$

4. From I let us single out the case when the function $\mathcal{L}(t,\tau) = \mathcal{K}(t,\tau)/(a(t) + b(t))$ is analytically continuable in D^+ both with respect to t and to τ. Then the solvability of equation $(1')$ in closed form acquires a simple operator interpretation. Let L be the integral operator with kernel $\mathcal{L}(t,\tau)$.

The operator

$$N\varphi = a\varphi + bS\varphi + (a+b)L\varphi$$

is the composition of the characteristic operator $aI + bS$ and the invertible operator $I + L$:

$$N = (aI + bS)(I + L).$$

Since $L^2 = 0$ (see the proof of the lemma), it follows that $(I + L)^{-1} = I - L$.

Let us note in conclusion that, analogously, the case can be considered in I-II when the kernel $\mathcal{K}(t,\tau)$ is meromorphic with respect to τ. In this case the kernel $\mathcal{K}(t,\tau)$ can be represented as the sum of a kernel analytic with respect to τ and a degenerate kernel. Equation $(1')$ is reduced to a Riemann problem of type (4) and to a linear algebraic system.

Rostov State University

Received
24 IV 1969

REFERENCES

1. N. I. Muskhelishvili, *Singular integral equations*, Moscow, 1962.

2. F. D. Gakhov, *Boundary value problems*, Moscow, 1963.

3. F. D. Gakhov, L. I. Chibrikova, Matem. sbornik, 35 (77), No. 3, 395 (1954).

4. L. I. Chibrikova, Uchen. zap. Kazansk. univ., 122, No. 3, 95 (1962).

5. S. G. Samko, Izv. vyssh. uchebn. zaved., Matematika, No. 1, 67 (1969).

6. J. R. Hatcher, Am. Math. Monthly, 63, No. 9, 651 (1956).

7. A. S. Peters, Comm. Pure and Appl. Math., 18, 129 (1965).

8. K. M. Case, J. Math. Phys., 7, No. 12, 2121 (1967).

Note: Figure translations are in progress. See original paper for figures.