ENDOMORPHISMS OF THE LATTICE OF CLOSED SUBGROUPS OF A TOPOLOGICAL GROUP

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Abstract

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ENDOMORPHISMS OF THE LATTICE OF CLOSED SUBGROUPS OF A TOPOLOGICAL GROUP

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The set $L(G)$ of all closed subgroups of a topological group G is a complete lattice (structure) with respect to intersection and the taking of the smallest closed subgroup $A \vee B$ containing $A, B \in L(G)$. For the case of a discrete group G, an extensive literature is devoted to the relations between the structure of G and $L(G)$ (see, for example, the monograph of M. Suzuki ([1])). Homomorphisms of $L(G)$ have been studied deeply, in particular (([1], Chapter III). The present note is devoted to analogous problems for locally compact topological groups. Namely, the question is studied under what conditions the mapping $X \to X \vee S$, where S is a proper closed subgroup of G, is an endomorphism of the lattice $L(G)$. A complete solution of this problem in the discrete case was given by G. Pappa ([2]), D. Higman ([3]), and S. Sato ([4]). Since

$$(X \cap Y) \vee S = (X \vee S) \cap (Y \vee S),$$

if $X \to X \vee S$ is an endomorphism, the question turns out to be closely connected with distributive relations between $\cap$ and $\vee$. The distributive law in $L(G)$ was studied by us in ([5]), where analogues were obtained of the classical results of O. Ore ([6]) on groups with a distributive lattice and on distributive pairs of subgroups for topological groups. All groups considered below are assumed to be locally compact, which is not stipulated in the formulations.

A pair A, B of elements of $L(G)$ is called **distributive** (dually distributive) if, for every $X \in L(G)$, the distributive law holds

$$(A \vee B) \cap X = (A \cap X) \vee (B \cap X)$$

(respectively, the dual equality). These notions, introduced by O. Ore, were studied in ([6,7,5]). We have obtained ([5]) a criterion for distributivity of a pair of closed subgroups, playing an important role in the proof of the following theorems. We give it here in the following form.

Theorem 1. *For a pair A, B of closed subgroups of a group G to be distributive, it is necessary that:*

(1) the connected compact subgroups of $A \vee B$ lie in $A \cap B$;

(2) all primary elements of $A \vee B$ lie in $A \cup B$;

(3) every pure element of $(A \vee B) \setminus (A \cup B)$ have finite and relatively prime orders with respect to A and B,

and it is sufficient that conditions (2) and (3) be fulfilled.

Recall that an element x is called **primary** with respect to a prime number p, or a p-element, if $\lim x^{p^n} = 1$ as $n \to \infty$. By $\Pi(M)$ is denoted the set of all prime numbers p for which the subset M of G contains a p-element $\neq 1$. An element x is **pure** if the cyclic subgroup $\{x\}$ generated by it is infinite and closed. A group G is **periodic** if it has no pure elements.

An element $S \in L(G)$ forming a distributive pair with every element of $L(G)$ is called **standard**. The notion of a standard element-

introduced in (8). It is also established there that if S is standard, then $X \to X \vee S$ is a homomorphism, and dually. It is easy to verify that if $N \in L(G)$ is simultaneously standard and dually standard, then N generates a distributive sublattice, i.e., is a neutral element in $L(G)$. The neutral elements in the lattice of a finite group were described by M. Suzuki (9) and G. Zappa (10), thereby solving problem 35 of G. Birkhoff (11). Below we shall obtain a far-reaching generalization of this result. A neutral element of $L(G)$ possessing a complement is called central. The following theorem describes the central elements, serves as a criterion for the reducibility of $L(G)$, and at the same time gives a criterion for the distributivity and dual distributivity of a pair of mutually prime subgroups. It contains Jones' s theorem ((1), p. 19), as well as Theorem 2 from (7).

Theorem 2. *Let A and B be proper closed subgroups of a group G such that $A \vee B = G$, $A \cap B = E$. Then the following conditions are equivalent:*

(1) *the pair A, B is distributive in $L(G)$;*

(2) *the pair A, B is dually distributive in $L(G)$;*

(3) $L(G) = L(A) \times L(B)$;

(4) *the group G is zero-dimensional, periodic, and decomposes into the topological direct product $G = A \times B$, with $\Pi(A) \cap \Pi(B) = \emptyset$.*

Suppose now that $X \to X \vee S$ is a proper endomorphism of the lattice $L(G)$, i.e., $S \neq G$ and $S \neq E$. Then the following five propositions hold, with the help of which our results are obtained. As in the discrete case ((1), p. 100), it is proved that:

1°. S is a normal subgroup of G.

2°. *If $H \in L(G)$ and the subgroup HS is closed, then the pair H, S is distributive.*

It follows from this that S is standard if S is compact or open. In particular, for discrete groups the concepts of a standard subgroup and of the upper kernel of the endomorphism $X \to X \vee S$ coincide (S. G. Ivanov). In the general case, the coincidence of these two concepts can be established only after a description of the endomorphisms of $L(G)$.

3°. *The factor group G/S is periodic.*

4°. *S is incident with the connected component G_0 of the identity in G.*

5°. *If in $G \setminus S$ there is a p-element x, then S is zero-dimensional, elementwise permutable with x, all p-elements of S lie in $\{\bar{x}\}$, and the group G is periodic.*

We give a partial converse to proposition 5°, which nevertheless contains Theorem III. 6 from (1).

Theorem 3. *A compact or open proper subgroup S of a zero-dimensional periodic group G is standard in $L(G)$ if and only if, for every p-element x of $G \setminus S$, all p-elements of S lie in $\{\bar{x}\}$, and x belongs to the centralizer of the subgroup S in G.*

By somewhat narrowing the class of periodic groups, one can obtain a sharper result. We shall call a group G inductively compact if every finite set of its elements lies in a compact subgroup (i.e., G is equal to the inductive limit of compact groups). In a zero-dimensional inductively compact group all topological Sylow Π-subgroups, for any set Π of prime numbers, turn out to be closed. By Q_p (J_p) we denote the additive group of all (all integral) p-adic numbers, and by D_p the discrete group of type p^∞. Let the group Q be generated by the elements $b, a_1, a_2, \dots, a_n, \dots$, with relations $a_n^2 = a_{n-1}$, $a_1^2 = 1$, $b^2 = a_1$, $b^{-1} a_n b = a_n^{-1}$; the non-Abelian subgroups of Q are called generalized quaternion groups.

Theorem 4. *Let G be a zero-dimensional inductively compact group. If $X \to X \vee S$ is a proper endomorphism of $L(G)$, then $G = H \times K$, $\Pi(H) \cap \Pi(K) = \emptyset$, $S = H \times A$, where A is a central subgroup of G lying in K, and for every $p \in \Pi(A)$ every Sylow p-subgroup of K is either cyclic*

or one of the types D_p, J_p, Q_p, or the discrete generalized quaternion group. Conversely, in a group of the indicated structure, S is a standard subgroup.

No less strong a restriction is the presence of a standard subgroup in an aperiodic or nonzero-dimensional group.

Theorem 5. Let G be an aperiodic zero-dimensional group. If $X \to X \vee S$ is a proper endomorphism of $L(G)$, then:

(1) S is an open normal subgroup in G;

(2) G/S is periodic and $\Pi(S) \cap \Pi(G/S) = \emptyset$;

(3) if $x \in G \setminus S$, then x is pure, and for every $s \in S$ there exists an integer n such that $x^n \in xS,\ x^n s = s x^n$;

(4) in G there are no mutually coprime discrete infinite cyclic subgroups.

Conversely, if G satisfies conditions (1)–(4), then S is standard in $L(G)$.

This theorem contains Theorem III.7 from ([1]) and is proved analogously. The following theorem has no analogues for discrete groups.

Theorem 6. Let the group G be nonzero-dimensional. If $X \to X \vee S$ is a proper endomorphism of $L(G)$, then G is periodic and belongs to one of the following types:

I. G_0 is isomorphic to the special unitary group $SU(2)$ of complex matrices of order two; $S = \{s\}$ coincides with the center of G_0, $s^2 = 1$; every 2-element of G has finite order modulo S.

II. G_0 is a one-dimensional compact group ("solenoid"); $S = \{s\} \subseteq G_0$, $s^2 = 1$, S is central in G; all Sylow 2-subgroups of G are infinite quaternion groups.

III. G_0 is a solenoid; S is a zero-dimensional subgroup of G_0, central in G; $\Pi(S) \cap \Pi(G/G_0) = \emptyset$.

Conversely, in groups of the indicated types, S is a standard subgroup.

With the aid of Theorems 4 and 6, for a sufficiently broad class of locally compact groups an analogue of Problem 35 from ([11]) is solved.

Theorem 7. Let the group G be compact mod G_0. $L(G)$ has a proper neutral element N if and only if G is compact and one of the following conditions is fulfilled:

(1) G_0 is a solenoid, $G = \{b\} \cdot C,\ b^4 = 1$, where C is the centralizer of G_0 in G, $N = \{b^2\} \subseteq G_0$;

(2) G_0 is a solenoid and lies in the center of G, N is a zero-dimensional subgroup in G_0, $\Pi(N) \cap \Pi(G/G_0) = \emptyset$;

(3) G is zero-dimensional and is equal to $H \times (B \ltimes M)$; $\Pi(H)$, $\Pi(B)$, and $\Pi(M)$ are pairwise disjoint; $N = H \times A$; A is central in G and $\Pi(A) = \Pi(B)$; B is a metabelian group, each primary component of which is either cyclic, or J_p, or a generalized quaternion group.

From item (3) of Theorem 7 follows the result of M. Suzuki and T. Tsappa ([1], Theorem III.12), on which our proof does not rely. In the proofs we used certain results of V. P. Platonov ([12]) and D. Lee ([13]).

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Note: Figure translations are in progress. See original paper for figures.