ON A GENERALIZATION OF CAUCHY' S FUNDAMENTAL THEOREM

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Abstract

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ON A GENERALIZATION OF CAUCHY' S FUNDAMENTAL THEOREM

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Consider an associative finite-dimensional algebra $\mathfrak{A}$ over the field of real numbers. Let $e_1, e_2, \dots, e_s$ be some basis of $\mathfrak{A}$.

An arc Γ in $\mathfrak{A}$ is called rectifiable if it can be given parametrically by

$$y(t) = \sum_{i=1}^{s} y_i(t) e_i, \qquad t_0 \leqslant t \leqslant t_1,$$

where the $y_i(t)$ are continuous and of bounded variation. Let $f(y)$ be a continuous mapping of $\mathfrak{A}$ into $\mathfrak{A}$. The sum

$$\sum_i f(y(\xi_i))(y(t_{i+1}^{(n)}) - y(t_i^{(n)})),$$

where ξ_i is an arbitrary point of $[t_{i+1}^{(n)}, t_i^{(n)}]$, as

$$\max_i (t_{i+1}^{(n)} - t_i^{(n)}) \to 0$$

tends to a limit independent of the sequences $\{t_i^{(n)}\}$. We shall call this limit, as usual, the integral of $f(y)$ and denote it by

$$\int_\Gamma f(y)\, dy.$$

The linear span of the elements $xy - yx = [x, y]$ for $x, y \in \mathfrak{A}$ is called the commutator of $\mathfrak{A}$ and is denoted by $[\mathfrak{A}]$. Let Γ be a rectifiable contour in $\mathfrak{A}$. Then the following holds.

Theorem 1. *For any integer $n \geqslant 0$,*

$$\int_\Gamma y^n\, dy \in [\mathfrak{A}].$$

Proof. As in the classical case ([1]), it is enough to prove Theorem 1 for a triangular contour. Let Γ be a triangular contour with vertices A, B, O. The vertex O lies at the zero of the algebra. Then

$$\int_\Gamma y^n\,dy = \int_{OA} y^n\,dy + \int_{AB} y^n\,dy + \int_{BO} y^n\,dy.$$

Put $\overrightarrow{OA} = a,\ \overrightarrow{AB} = b,\ \overrightarrow{BO} = d.$

Clearly, $a + b + d = 0$. Denote by $M(n-m, m)$ the set of monomials α in the expansion of $(a+b)^n$ that contain m factors b and $n-m$ factors a, and put

$$Q(n-m, m) = \sum_{\alpha \in M(n-m,m)} \alpha.$$

Then

$$\int_{OA} y^n\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{i}{N}a\right)^n \left(\frac{1}{N}a\right) = a^{n+1} \int_0^1 t^n\,dt = \frac{1}{n+1}a^{n+1},$$

$$\int_{AB} y^n\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} \left(a + \frac{i}{N}b\right)^n \left(\frac{1}{N}b\right) = \left[\sum_{m=0}^{n} \frac{1}{m+1} Q(n-m, m) b\right],$$

$$\int_{BO} y^n\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} \left\{(a+b) + \frac{i}{N}d\right\}^n \left(\frac{1}{N}d\right) = -(a+b)^{n+1} \lim_{N\to\infty} \sum_{i=1}^{N} \left(1 - \frac{i}{N}\right)^n \frac{1}{N}$$

$$= -(a+b)^{n+1} \int_0^1 (1-t)^n\,dt = -\frac{1}{n+1}(a+b)^{n+1}.$$

After simplification we obtain

$$\int_\Gamma y^n\,dy = \sum_{m=0}^{n-1} \left\{\frac{1}{m+1} Q(n-m, m) b - \frac{1}{n+1} Q(n-m, m+1)\right\}.$$

We shall show that the difference

$$\rho = (n+1)Q(n-m, m)b - (m+1)Q(n-m, m+1) \in [\mathfrak{A}], \qquad m = 0, \dots, n-1.$$

The number of terms of each sum

$Q(n-m, m)b$ and $Q(n-m, m+1)$ are equal respectively to $\binom{n}{m}$ and $\binom{n+1}{m+1}$. Therefore the number of monomials in $(n+1)Q(n-m, m)b$ and in $(m+1)Q(n-m, m+1)$ is the same. We shall match the monomials from these sums so that their difference is a commutator.

Let α be a monomial. Then α_i is equal to the product of the i first factors of α in the order in which they occur, and β_j, respectively, to the product of the j last factors, i.e. $\alpha = \alpha_i \beta_{n-i}$, if α has n factors. Let, further,

$$R_i(\alpha) = \alpha_i b \beta_{n-i}, \qquad R_i(\alpha + \alpha' + \ldots) = R_i(\alpha) + R_i(\alpha') + \ldots .$$

Lemma 1.

$$R(Q(n,m)) = bQ(n,m) + \sum_{i=1}^{n+m-1} R_i(Q(n,m)) + Q(n,m)b =$$

$$= (m+1)Q(n,m+1). \tag{1}$$

Proof. If (1) holds for $Q(n-1,m)$ and $Q(n,m-1)$, then it also holds for $Q(n,m)$. Indeed,

$$Q(n,m) = Q(n-1,m)a + Q(n,m-1)b. \tag{2}$$

Then

$$R(Q(n,m)) = R(Q(n-1,m)a) + R(Q(n,m-1)b).$$

But

$$R(Q(n-1,m)a) = R(Q(n-1,m))a + Q(n-1,m)ab$$

and

$$R(Q(n,m-1)b) = R(Q(n,m-1))b + Q(n,m-1)b^2.$$

Taking now (1) and (2) into account, we obtain

$$\begin{aligned} R(Q(n,m)) &= (m+1)Q(n-1,m+1)a + mQ(n,m)b \\ &\quad + [Q(n-1,m)a + Q(n,m-1)b]b \\ &= (m+1)Q(n-1,m+1)a + mQ(n,m)b + Q(n,m)b \\ &= (m+1)[Q(n-1,m+1)a + Q(n,m)b] \\ &= (m+1)Q(n,m+1). \end{aligned}$$

To prove the lemma it remains to show that (1) holds for $Q(0,m)$ and $Q(n,0)$ for arbitrary n and m. But from the definition of R we have

$$R(Q(0,m)) = (m+1)b^{m+1} = (m+1)Q(0,m+1),$$

$$R(Q(n,0)) = ba^n + aba^{n+1} + \ldots + a^n b = Q(n,1).$$

Put now $\pi_j(\alpha) = \beta_{n-j}\alpha_j$ for $\alpha \in M(n-m,m) = M$,

$$\pi_j(\alpha + \alpha' + \ldots) = \pi_j(\alpha) + \pi_j(\alpha') + \ldots .$$

Since from $\pi_j(\alpha) = \pi_j(\alpha')$ it follows that $\alpha = \alpha'$ and conversely (here equality is understood in the sense of coincidence of the factors in the same positions), we have

$$\pi_j\left(\sum_{\alpha\in M}\alpha\right)=\sum_{\alpha\in M}\pi_j(\alpha)=\sum_{\alpha\in M}\beta_{n-j}\alpha_j=Q(n-m,m).$$

Therefore

$$\pi_j(Q(n-m,m)b)=\sum_{\alpha\in M}\pi_j(\alpha b)=\sum_{\alpha\in M}\beta_{n-j}b\alpha_j=$$

$$=R_{n-j}\left(\sum_{\alpha\in M}\beta_{n-j}\alpha_j\right)=R_{n-j}(Q(m-m,m)). \qquad (3)$$

By virtue of the lemma and (3) we have

$$(m+1)Q(n-m,m)=bQ(n-m,m)+\sum_{i=1}^{n-1}R_i(Q(n-m,m))+$$

$$+Q(n-m,m)b=bQ(n-m,m)+\sum_{j=1}^{n-1}\pi_j(Q(n-m,m)b)+Q(n-m,m)b.$$

Consequently

$$\varphi=Q(n-m,m)b-bQ(n-m,m)+\sum_{j=1}^{n-1}\{Q(n-m,m)b-\pi_j(Q(n-m,m)b)\}=$$

$$=\sum_{\alpha\in M}[\alpha,b]+\sum_{j=1}^{n-1}\left(\sum_{\alpha\in M}(\alpha b-\pi_j(\alpha b))\right).$$

Since from the definition of π_j it follows that

$$\alpha b-\pi_j(\alpha b)=[\alpha_j,\beta_{n-j}b],$$

then

$$\rho=\sum_{\alpha\in M}[\alpha,b]+\sum_{j=1}^{n-1}\left(\sum_{\alpha\in M}[\alpha_j,\beta_{n-j}b]\right)\in[\mathfrak{A}].$$

The theorem is proved.

Considering $[\mathfrak{A}]$ as a linear subspace of $\mathfrak{A}$, define B, $B \perp [\mathfrak{A}]$. If $L = a_1 e_1 + \cdots + a_{se} s \perp [\mathfrak{A}]$, then for $x = x_1 e_1 + \cdots + x_{se} s \in [\mathfrak{A}]$ we have $L(x) = a_1 x_1 + \cdots + a_{sx} s = 0$. The totality of such forms has dimension equal to the dimension of B. According to the theorem, the integrals $\int_a^b y^n \, dy$, taken along different paths from a to b, differ by an element of $[\mathfrak{A}]$. Therefore the following generalization of the fundamental theorem of Cauchy holds.

Theorem 2. *The value*

$$L\left(\int_a^b y^n \, dy\right)$$

does not depend on the path connecting the points a and b.

Suppose now that $\mathfrak{A}$ has a unit. We shall assume that the basis element e_1 is the unit of $\mathfrak{A}$.

Theorem 3.

$$L\left(\int_\Gamma y^{-1} dy\right) = 0,$$

if the contour Γ belongs to some simply connected domain D in which y^{-1} is defined.

Proof. Since D is simply connected, for arbitrary $d > 0$ there exist in D contours Γ_i of diameter $< d$ such that

$$\int_\Gamma y^{-1} dy = \sum_i \int_{\Gamma_i} y^{-1} dy.$$

Let $y_i \in D$ be such that $|x - y_i| \leq d$ for $x \in \Gamma_i$, and let d be chosen so that from $|y| < d$ it follows that

$$(e_1 + y)^{-1} = \sum_{n=0}^{\infty} (-y)^n.$$

Then

$$\int_{\Gamma_i} y^{-1} dy = \int_{\Gamma_i - y_i} (y_i + z)^{-1} dz = \int_{\Gamma_i - y_i} \left(y_i (e_1 + y_i^{-1} z)\right)^{-1} dz = \int_{y_i^{-1}(\Gamma_i - y_i)} (e_1 + y)^{-1} dy.$$

Since for $|y| < d$

$$(e_1 + y)^{-1} = \sum_{n=0}^{\infty} (-y)^n,$$

Theorem 2 is fulfilled for it. Theorem 3 is proved.

The function

$$\int_{e_1}^x y^{-1} dy$$

has an application in probability theory on finite groups.

Lemma 2.

$$L\left(\int_{e_1}^{x} y^{-1}dy\right) = \varphi(x)$$

is a one-dimensional representation of a neighborhood of the unit of $\mathfrak{A}$.

Indeed, since

$$\int_{x_1}^{x_1x_2} y^{-1}dy = \int_{e_1}^{x_2} (x_1y)^{-1}\,d(x_1y) = \int_{e_1}^{x_2} y^{-1}dy,$$

we have

$$\varphi(x_1x_2)L\left(\int_{e_1}^{x_1x_2} y^{-1}dy\right) = L\left(\int_{e_1}^{x_1} y^{-1}dy + \int_{x_1}^{x_1x_2} y^{-1}dy\right) = \varphi(x_1) + \varphi(x_2).$$

Let now $\mathfrak{A}$ be the group algebra of some group G. Then one may assume that the basis $e_1, e_2, \dots, e_s$ is the group G. In this case compute

$$\varphi_j(x) = L_j\left(\int_{e_1}^{x} y^{-1}dy\right),$$

where L_j, $j = 1, \dots, r$, is a basis of linear forms,

corresponding to some basis $\{\gamma^{(j)}\}$ of the orthogonal complement to $\mathfrak{A}$. It is not difficult to show that in the case of a group algebra one may take for $\{\gamma^{(j)}\}$ vectors consisting of zeros and ones, the ones in $\gamma^{(j)}$ occurring only in those places α for which e_α belongs to the j-th class of conjugate elements of G. Here a fixed numbering of these classes is meant; denote them by T_j, $j = 1, \dots, r$. Then

$$\varphi_i(x) = \sum_{j=1}^{r} \frac{h_j}{s}\chi_i(T_j^{-1}) \ln|R_i|, \tag{4}$$

where $|R_i|$ is the determinant of the i-th irreducible part of the group matrix of the element x; χ_i is the character of the i-th irreducible representation; h_j is the number of elements of T_j; s is the order of the group G.

Let $y(t)$, $t \geqslant 0$, $y(0) = e_1$, be a continuous curve in $\mathfrak{A}$ such that for each t there exists $y^{-1}(t)$, and for the elements $y^{-1}(t_1)y(t_2)$, denoted by $y(t_1, t_2) = p_1(t_1, t_2)e_1 + \dots + p_s(t_1, t_2)e_s$, one has $p_i(t_1, t_2) \geqslant 0$,

$$\sum_{i=1}^{s} p_i(t_1, t_2) = 1.$$

Then $y(t_1, t_2)$ may be regarded as a probability distribution on the group G. From the distributions $y(t)$ one can construct a process with independent increments on the group G such that the distribution of the increment of the process on the interval $[t_1, t_2]$ is equal to $y(t_1, t_2)$. The number of jumps of such a process is finite almost everywhere. Using expression (1) from (2), one can show that the curve $y(t)$ on $[0, t]$ is rectifiable and, for

$$\Delta_n = \max_i (t_{i+1}^{(n)} - t_i^{(n)}) \to 0, \qquad \sum_{i=1}^{n-1} p_j(t_i^{(n)}, t_{i+1}^{(n)}) \to m_j(t) \geqslant 0, \quad j \geqslant 2,$$

$$\sum_{i=1}^{n-1} \{p_i(t_i^{(n)}, t_{i+1}^{(n)}) - 1\} \to m_1(t).$$

The functions $m_j(t)$, $j \geqslant 2$, are nondecreasing and continuous. From (3), the number of jumps of the process under consideration on the interval $[t_1, t_2]$ into the set $A \subset G/e_1$ is distributed according to the Poisson law. The parameter of this Poisson distribution is equal to

$$\sum_{i,\, e_i \in A} m_i(t_2) - m_i(t_1).$$

Let us integrate the function y_{n-1}^{-1} along the curve $y(t)$. We have

$$L_j \left(\int_{e_1}^{y(t)} y^{-1}\, dy \right) = L_j \left(\lim_{\Delta_n \to 0} \sum_{i=1}^{n-1} y^{-1}(t_i^{(n)}) (y(t_{i+1}^{(n)}) - y(t_i^{(n)})) \right) =$$

$$= L_j \left(\lim_{\Delta_n \to 0} \sum_{i=0}^{n-1} y^{-1}(t_i^{(n)}) y(t_i^{(n)}) (y(t_i^{(n)}, t_{i+1}^{(n)}) - 1) \right) = \sum_{i,\, e_i \in T_j} m_i(t).$$

By Theorem 3 this integral does not depend on the path of integration. Therefore the quantity

$$\sum_{i,\, e_i \in T_j} m_i(t)$$

will be the same for different curves y_1, y_2, provided $y_1(t) = y_2(t)$.

We shall say that a set $A \subset G \setminus e_1$ is invariant if the distribution of jumps into this set on any interval $[t_1, t_2]$ depends only on the distribution of the increment of the process, i.e., on $y(t_1, t_2)$. Then, on the basis of what has been said, the following holds.

Theorem 4. *A set $A \subseteq G \setminus e_1$ is invariant if and only if it is the union of some number of conjugacy classes T_j.*

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Note: Figure translations are in progress. See original paper for figures.