SOME SOLUTIONS OF AN INHOMOGENEOUS DIFFUSION EQUATION

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Fig. 1

Figure 1: Fig. 1

Abstract

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SOME SOLUTIONS OF AN INHOMOGENEOUS DIFFUSION EQUATION

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We consider the following problem, close to those studied in works ([1–9]): to find a bounded solution of the inhomogeneous diffusion equation

$$\partial u/\partial t = a^2 \partial^2 u/\partial x^2 + F[u(\bar{x}, t)], \tag{1}$$

periodic, or such that the initial function $u(x, 0)$ over a previously unknown time interval T increases by a factor k $(k > 1)$, with the boundary condition

$$u(0, t) = 0 \tag{2}$$

in the semi-infinite domain $0 < x < \infty$. Here the function $F(u)$ has the form shown in Fig. 1, i.e. $F[u(\bar{x}, t)] = c$ $(c > 0)$, if $u(\bar{x}, t) < u_*$, until $u(\bar{x}, t) = u_*$; if at $t = T_1$, $u(\bar{x}, T_1) = u_*$, then for $t > T_1$, $F[u(\bar{x}, t)] = -d$ $(d > 0)$, if $u(\bar{x}, t) > u_{**}$ $(u_{**} < u_*)$, until $u(\bar{x}, t) = u_{**}$ at $t = T$. In doing this we shall assume that

Fig. 1

$$u(x, T) = k\omega(x) \quad (\omega(x) = u(x, 0);\; k \geq 1), \tag{3}$$

so that it must be

$$u_* = ku(\bar{x}, 0).$$

Thus, the formulation of the problem is analogous to that given in works ([8,9]).

Below, for $k \geq 1$, a solution of this problem is found for large values of T/T_1 and small d/c, under the assumption that the quantities c, d, T_1, and T are related by the relation $(c+d)T_1 = dT$. A solution of this problem is also obtained under the condition $\bar{x} \gg 2a\sqrt{T}$, and, if $k=1$, then $(c+d)T_1 = dT$.

As an example of the problem described by equation (1) with conditions (2), (3), one may consider the following problem of filtration theory in a hydraulic formulation: let groundwater be situated in the region $0 \le x < \infty$; let $u(x,t)$ denote the groundwater level; at the section $x=0$ there is a wall of a canal in which a constant, in particular zero, liquid level is maintained; at the section $x=\bar{x}$ $(0<\bar{x}<\infty)$ the height of the groundwater mound is measured; as soon as the height of the mound reaches the value u_*, uniform irrigation carried out with infiltration intensity (taking evaporation into account) mc is stopped, and only evaporation with intensity md occurs; when the height of the mound has decreased to the value u_{**}, irrigation is again begun with the previous intensity; find the solution of the problem on the spreading of the groundwater mound. Here m is the active porosity of the soil, $a^2 = k\bar{u}/m$, k is the filtration coefficient, and $\bar{u}$ is the mean depth of the flow.

Let, for $0<t<T_1$, $F[u(\bar{x},t)] = c$, and for $T_1<t<T$, $F[u(\bar{x},t)] = -d$; at $t=T$ relation (3) is satisfied. We shall find the solution $u(x,t)$ of equation (1) with boundary condition (2).

The function $u(x,t)$, expressed in terms of the as yet unknown function $\omega(x)$, is determined by the formulas

$$\begin{aligned} &u(x,t) = v(c,x,t) + w(x,t) && (0 \le t \le T_1), \\ &u(x,t) = v(c,x,t) - v(c,x,t-T_1) - v(d,x,t-T_1) + w(x,t) && (T_1 \le t \le T), \end{aligned} \tag{4}$$

where

$$v(c,x,t) = c\left[-\frac{x^2}{2a^2} + \left(t+\frac{x^2}{2a^2}\right)\Phi\left(\frac{x}{2a\sqrt{t}}\right) + \frac{x\sqrt{t}}{a\sqrt{\pi}}\exp\left(-\frac{x^2}{4a^2t}\right)\right],$$

$$w(x,t) = \frac{\exp(-x^2/4a^2t)}{a\sqrt{\pi t}}\int_0^\infty \exp\left(-\frac{\xi^2}{4a^2t}\right)\operatorname{sh}\frac{x\xi}{2a^2t}\,\omega(\xi)\,d\xi, \tag{5}$$

$$\Phi(y) = \frac{2}{\sqrt{\pi}}\int_0^y \exp(-S^2)\,dS.$$

Condition (3) gives the integral equation

$$\varphi(x) = \psi(x) + \lambda\int_0^\infty K(x,\eta)\varphi(\eta)\,d\eta \tag{6}$$

for finding the function $\varphi(x)$. Here it is assumed that

$$\varphi(x) = \omega(x), \qquad \psi(x) = f(x) \qquad (k = 1),$$

$$\varphi(x) = -g(x) + \omega(x), \qquad g(x) = D\left[1 + \sum_{m=1}^{\infty} \frac{1}{k^m} \Phi\left(\frac{x}{2a\sqrt{mT}}\right)\right], \tag{7}$$

$$\psi(x) = -D + f(x)/k \qquad (k > 1),$$

where

$$f(x) = \gamma x^2[1-\Phi(\alpha x)] - \zeta x^2[1-\Phi(\beta x)] + \mu\Phi(\alpha x) + \chi\Phi(\beta x) - \delta x e^{-\alpha^2 x^2} + \varepsilon x e^{-\beta^2 x^2}, \tag{8}$$

$$\Bigg(\gamma = \frac{c+d}{2a^2},\ \zeta = \frac{c}{2a^2},\ \alpha = \frac{1}{2a\sqrt{T-T_1}},\ \beta = \frac{1}{2a\sqrt{T}},$$

$$\mu = (c+d)(T_1 - T),\ \chi = cT,\ \delta = \frac{(c+d)\sqrt{T-T_1}}{a\sqrt{\pi}},$$

$$\varepsilon = \frac{c\sqrt{T}}{a\sqrt{\pi}},$$

$$D = [cT_1 + d(T_1 - T)]/k, \quad f(\infty) = Dk);$$

$$\lambda = \frac{1}{ka\sqrt{\pi T}}, \qquad K(x,\eta,T) = \exp\left[-\frac{x^2+\eta^2}{4a^2T}\right] \operatorname{sh} \frac{x\eta}{2a^2T}. \tag{9}$$

Consider the homogeneous integral equation corresponding to (6) for the case $k = 1$

$$\varphi(x) = \frac{1}{a\sqrt{\pi T}} \int_0^\infty \exp\left[-\frac{x^2+\eta^2}{4a^2T}\right] \operatorname{sh} \frac{x\eta}{2a^2T}\, \varphi(\eta)\, d\eta. \tag{10}$$

As is known, for (10) the relation

$$\int_0^\infty \eta \exp\left(-\frac{\eta^2}{4a^2T}\right) \operatorname{sh} \frac{x\eta}{2a^2T}\, d\eta = a\sqrt{\pi T}\, x \exp \frac{x^2}{4a^2T} \tag{11}$$

holds.

In consequence of (11), it is clear that the integral equation (10), for any B, has the solution $\varphi(x) = Bx$. Formulas (4), (5), and (7) show that, for boundedness of the solution $u(x,t)$ of the problem under consideration, the condition $B = 0$ must be satisfied.

Introducing the notation

$$R(x,\eta,T) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{mk^{m-1}}} \exp\left[-\frac{x^2+\eta^2}{4a^2mT}\right] \operatorname{sh}\frac{x\eta}{2a^2mT}, \tag{12}$$

one can see that, if in formula (8) $\psi(x) \to 0$ as $x \to \infty$, the solution $\varphi(x)$ of integral equation (6) is represented in the form

$$\varphi(x) = \psi(x) + \frac{1}{ka\sqrt{\pi T}} \int_0^{\infty} R(x,\eta,T)\psi(\eta)\, d\eta. \tag{13}$$

It follows from formulas (7) and (8) that for $k = 1$, $\psi(\infty) = D$, and for $k \neq 1$, $\psi(\infty) = 0$. For large m the general term of the series (12) has the form

$$\frac{1}{2a^2T}\frac{x\eta\psi(\eta)}{k^{m-1}m^{3/2}} + \cdots .$$

In the case $k = 1$, solution (13), by virtue of (8), exists only when $D = 0$, which gives the condition

$$cT_1 = d(T - T_1). \tag{14}$$

Thus, we have obtained the following result: the problem sought has a solution defined by formula (13), for $k > 1$, for arbitrary values of the constants c, d, T_1, and T; for $k = 1$, the solution of the problem exists only for certain values of c, d, T_1, and T, namely those connected by equation (14).

The graphs of the function $f(x)$ for various values of the parameters c, d, T_1, and T, connected by condition (14), are presented in Fig. 2. Here the solid curve corresponds to the values $c = 1$, $d = 1/99$, $T_1 = 1$, and $T = 100$, and the dashed curve to the values $c = 1$, $d = 1$, $T_1 = 1$, $T = 2$ ($a = 1$).

The constants T_1 and T are connected with the quantities u_* and u_{**} by the relations

$$u(\bar{x}, T_1) = u_*, \qquad ku(\bar{x}, 0) = u_{**}. \tag{15}$$

In the case $k = 1$, the period of self-oscillation T is determined from equation (14), which must therefore be consistent with the equation $ku(\bar{x}, 0) = u_{**}$.

Fig. 2

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

Fig. 3

We now let, in formulas (4) and (5), the quantity T/T_1 tend to infinity and d/c tend to zero in such a way that condition (14) is preserved. From (8), (12), and (13) it is clear that $f(x) = \varphi(x) = \omega(x) = \psi(x) = 0$, and formulas (4) and (5) pass into the following:

$$\begin{aligned} &u(x,t) = v(c,x,t) && (0 \le t \le T_1), \\ &u(x,t) = v(c,x,t) - v(c,x,t-T_1) && (T_1 \le t \le T). \end{aligned} \tag{16}$$

Relations (16) give a solution of problem (1)–(3) for which

$$\lim_{t\to\infty} u(x,t) = ku(x,0) = k\omega(x) = 0.$$

Formulas (16) and (5) show that, for any $\bar{x}$, $\partial u(\bar{x},t)/\partial t > 0$ for $0 < t < T_1$, and $\partial u(\bar{x},t)/\partial t < 0$ for $T_1 < t < T$.

The graph of the function $u = u(\bar{x},t)/c$, given by relations (16), for the case $\bar{x} = 31.622777,\ d = 0,\ T_1 = 400,\ T = \infty,\ a = 1$, is shown in Fig. 3.

Let now the quantity T/T_1 be large, and d/c small, with condition (14) satisfied. Expanding the functions $\psi(x)$ and $R(x,\eta,T)$, given by formulas (7), (8), and (12), in series in powers of $T^{-1/2}$ and T^{-1}, respectively, and substituting these series into equation (13), we find an asymptotic representation of the function $\varphi(x)$ for large values of T. The first term of this expansion has the form

$$\varphi(x) = \frac{\nu}{\sqrt{T}}\, x + \cdots,$$

where

$$\nu = \frac{cT_1}{a\sqrt{\pi}}(-1+S), \qquad S = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 1\cdot 3 \ldots (2n+1)\sigma(n+{}^3/2)}{2^{n+1} n!(n+2)}.$$

Here $\sigma(s)$ denotes the series convergent for $k \ge 1$,

$$\sigma(s)=\sum_{m=1}^{\infty}\frac{1}{k^m m^s},$$

which for $k=1$ becomes the Riemann zeta function $\zeta(s)$ ([11]). It is easy to see that the series S entering the expression for the quantity ν also converges.

Relations (4) and (5) show that, in the case $\bar{x}\gg 2a\sqrt{T}$, and also for sufficiently large T/T_1 and sufficiently small d/c, if $\bar{x}\leq 2a\sqrt{T}$, the conditions $\partial u(\bar{x},t)/\partial t>0$ for $0<t<T_1$, $\partial u(\bar{x},t)/\partial t<0$ for $T_1<t<T$ are also satisfied, and the quantities T_1 and T can be found from relations (14) and (15).

Let us now consider problem (1)–(3) for $k=1$ on the finite interval $0<x<l$. The function $u(x,0)=\varphi(x)$ is determined by formulas (12)–(13), where $R(x,\eta,T)$ is replaced by

$$R(x,\eta,T)+\sum_{s=1}^{\infty}\{R(x-2sl,\eta,T)+R(x+2sl,\eta,T)\},$$

and

$$\psi(x)=\sum_{n=-\infty}^{\infty}[2\Lambda(\alpha_n)-\Lambda(\beta_n)+\Lambda(\gamma_n)],$$

$$\Lambda(\alpha)=\alpha^2\left[cM(\alpha/\sqrt{T})-(c+d)M(\alpha/\sqrt{T-T_1})\right],$$

$$M(y)=\exp(-y^2)/\sqrt{\pi}y+[1+1/2y^2]\Phi(y)-1,$$

$$\alpha_n=(x-2nl)/2a,\qquad \beta_n=\alpha_n+l/2a,\qquad \gamma_n=-\alpha_n+l/2a.$$

From the results obtained in work ([9]) there follows the existence of a solution of the problem under consideration for $k=1$ in the finite domain $0\leq x\leq l$.

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Note: Figure translations are in progress. See original paper for figures.