# AN EXAMPLE OF A CONDENSING OPERATOR IN THE THEORY OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT OF NEUTRAL TYPE

MATHEMATICS

1969

SovietRxiv

## Abstract

## Full Text

UDC 517.92

*MATHEMATICS*

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# AN EXAMPLE OF A CONDENSING OPERATOR IN THE THEORY OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT OF NEUTRAL TYPE

*(Presented by Academician I. N. Vekua on 22 XI 1968)*

In the present article the question of the existence of solutions of an equation of neutral type is investigated

$$x'(t) = f[t, x(t), x(t-h_1(t)), x'(t-h_2(t))]. \tag{1}$$

If the function $f(t,x,y,z)$ satisfies the Lipschitz condition with respect to the variables $x, y, z$, with constants $k_x, k_y, k_z$, respectively, and moreover $k_z < 1$, then under small additional assumptions the question of the existence of solutions is easily reduced to the contraction mapping principle (see ([1])). Here we shall abandon the Lipschitz condition with respect to the variables $x, y$. To prove the existence theorem in this case, the fixed-point principle for condensing operators ([2]) will be applied. As P. P. Zabreiko observed, this problem can be reduced in another way to Schauder' s principle.

1. Let us recall the fixed-point principle from ([2]). Let $E$ be a Banach space and $\Omega \subseteq E$. The numerical set $Q(\Omega)$ is defined as follows: a positive number $\varepsilon$ belongs to $Q(\Omega)$ if and only if the set $\Omega$ has a finite $\varepsilon$-net. The measure of noncompactness of the set $\Omega$ is called the quantity

$$\chi(\Omega) = \inf Q(\Omega).$$

(See also ([3–5]).) It is clear that if in the definition of the set $Q(\Omega)$ the finite $\varepsilon$-net is replaced by a compact one, then the quantity $\chi(\Omega)$ will not change. Let $T \subseteq E$. An operator $A: T \to E$ is called condensing if it is continuous on $T$ and, for any noncompact set $\Omega$, the inequality

$$\chi[A(\Omega)] < \chi(\Omega)$$

holds.

**Theorem 1.** *If a condensing operator $A$ maps a convex closed bounded set $T$ into itself, then it has at least one fixed point in $T$.*

2. We shall consider (1) together with the initial condition

$$x(t) = x_0(t) \qquad (-h < t \leq 0), \tag{1'}$$

where $x_0(t)$ is a fixed function defined on the half-interval (finite or infinite) $(-h, 0]$. By a solution of problem (1)–(1′) we shall understand a function $x(t)$ $(-h < t \leq H]$, satisfying the initial condition (1′) and the following three requirements: a) $x(t)$ is continuous on $(-h, H]$; b) $x'(t)$ exists almost everywhere on $(-h, H]$ and is summable to the power $p \geq 1$; c) almost everywhere on $[0, H]$

$$x'(t) = f[t, x(t), x(t - h_1(t)), x'(t - h_2(t))].$$

We shall denote by $\mathcal{E}(0, H)$ the set of functions continuous on $[0, H]$ and possessing a derivative summable to the power $p$; this set becomes a Banach space with the natural linear operations if one sets $\|x\|_{\mathcal{E}} = \|x\|_C + \|x'\|_{\mathcal{L}_p}$* ([1]). For any function

* $\mathcal{E}(0, H)$ is the Sobolev space $W_p^1$ up to an equivalent norm.

for $x(t) \in \mathcal{E}(0, H)$ we put

$$\tilde{x}(t) = \begin{cases} x_0(t), & -h < t < 0, \\ x(t), & 0 \leq t \leq H. \end{cases}$$

Along with problem (1)—(1′), let us consider the operator equation in the space $\mathcal{E}(0, H)$:

$$y = Iy, \tag{2}$$

where the operator $I$ is defined by the formula

$$Iy(t) = x_0 + \int_0^t f\left[s, y(s), \tilde{y}(s - h_1(s)), \tilde{y}'(s - h_2(s))\right] ds$$

$$(x_0 = x_0(0)).$$

It is not hard to verify that if the function $x_0(t)$ is continuous, and its derivative is summable with power $p$, then equation (2) is equivalent to problem (1)—(1′)

in the following sense: if $x(t)$ is a solution of problem (1)—(1′), then its restriction $y(t)$ to the interval $[0, H]$ is a solution of equation (2), and conversely, if $y(t)$ is a solution of equation (2), then the function $x(t) = \tilde{y}(t)$ is a solution of problem (1)—(1′).

3. Let us establish some properties of the operator $I$.

**Lemma 1.** Let $\mathcal{E}_0$ be the set of functions from $\mathcal{E}(0, H)$ satisfying the condition $x(0) = x_0$. Suppose that the functions $x_0(t)$, $h_1(t)$, $h_2(t)$, and $f(t, x, y, z)$ satisfy the following requirements: (I) $x_0(t)$ is continuous and bounded, and $x_0'(t)$ is summable with power $p$ on $(-h, 0]$; (II) $-H+t \le h_i(t) \le h+t$ $(i = 1, 2;\ 0 \le t \le H)$; (III) $h_1(t)$, $h_2(t)$ are measurable on $[0, H]$; (IV) the function $q(t) = t-h_2(t)$ is such that: a) the inverse image of every set of measure zero is measurable and b) for any measurable set $E \subseteq [0, H]$ satisfying the condition $q(E) \subseteq [0, H]$, the inequality $\mu E \le r\mu q(E)$ holds (the number $r$ does not depend on $E$); (V) $f(t, x, y, z)$ is defined for $0 \le t \le H$ and arbitrary real $x, y, z$; (VI) $f(t, x, y, z)$ is measurable in $t$ for any fixed $x, y, z$; (VII) $f(t, x, y, z)$ is continuous jointly in $x, y$ for fixed $t, z$; (VIII) $f(t, x, y, z)$ satisfies a Lipschitz condition in $z$:

$$|f(t, x, y, z_1) - f(t, x, y, z_2)| \le k|z_1 - z_2|;$$

(IX) for any $R > 0$ one can specify a function $m_R(t) \in \mathcal{L}_p(0, H)$ such that

$$|f(t, x, y, z)| \le m_R(t)$$

$$(0 \le t \le H;\ |x - x_0|,\ |y - x_0| \le R;\ -\infty < z < \infty).$$

Then the operator $I$ maps $\mathcal{E}_0$ into $\mathcal{E}_0$ and is continuous.

**Proof.** It is not hard to show that the operator $I$ maps $\mathcal{E}_0$ into $\mathcal{E}_0$. We shall show that it is continuous.

Let $y_n, y \in \mathcal{E}_0$ and $y_n \to y$, i.e. $\|y_n - y\|_C \to 0$, $\|y_n' - y'\|_{\mathcal{L}_p} \to 0$. Let $z_n = Iy_n$, $z = Iy$. Put

$$u_n(t) = f\big[t, y_n(t), \tilde{y}_n(t - h_1(t)), \tilde{y}'(t - h_2(t))\big]$$

(this function, as is not hard to see, also belongs to $\mathcal{L}_p(0, H)$). Then

$$\|z_n' - z'\|_{\mathcal{L}_p} \le \|z_n' - u_n\|_{\mathcal{L}_p} + \|u_n - z'\|_{\mathcal{L}_p}. \tag{3}$$

The second summand tends to zero by conditions (VII), (IX), and Lebesgue’ s theorem on passage to the limit under the integral sign. We estimate the first summand with the aid of the Lipschitz condition:

$$\|z_n' - u_n\|_{\mathcal{L}_p} \leq k \left\{ \int_0^H |\tilde{y}_n'(s - h_2(s)) - \tilde{y}'(s - h_2(s))|^p \, ds \right\}^{1/p}. \quad (4)$$

Put $E = \{s \mid 0 \leq s \leq H,\ 0 \leq q(s) \leq H\}$. Obviously,

$$\int_0^H |\tilde{y}_n'(s - h_2(s)) - \tilde{y}'(s - h_2(s))|^p \, ds \leq \int_E |y_n'(s - h_2(s)) - y'(s - h_2(s))|^p \, ds, \quad (5)$$

since for $s \notin E$

$$\tilde{y}_n'(s - h_2(s)) = \tilde{y}'(s - h_2(s)) = x_0'(s - h_2(s)).$$

Further, from condition (IV) there follows the inequality

$$\int_E |y_n'(s - h_2(s)) - y'(s - h_2(s))|^p \, ds \leq r \int_{q(E)} |y_n'(s) - y'(s)|^p \, ds. \quad (6)$$

To prove this fact, introduce the notation $\psi(t) = |y_n'(t) - y'(t)|^p$ and consider the set

$$E_i = \{s \in E \mid a_{i-1} \leq \psi(s - h_2(s)) < a_i\},$$

where $a_{i-1}, a_i$ are constants. Obviously,

$$q(E_i) = \{s \in q(E) \mid a_{i-1} \leq \psi(s) < a_i\}.$$

Taking condition (IV) into account, we obtain the following relation for the integral sums of the integrals occurring in (6): $\sigma_1 \leq r\sigma_2$. Hence inequality (6) is obtained by passage to the limit. From (4), (5), and (6) we obtain:

$$\|z_n' - u_n\|_{\mathcal{L}_p} \leq k r^{1/p} \|y_n' - y'\|_{\mathcal{L}_p}.$$

Consequently, the first term in (3) also tends to zero. Thus,

$$\|z_n - z'\|_{\mathcal{L}_p} \to 0 \quad \text{as } n \to \infty. \quad (7)$$

We now estimate the quantity $\|z_n - z\|_C$:

$$\|z_n - z\|_C = \max_{0 \leq t \leq H} |z_n(t) - z(t)| \leq$$

$$\leq \int_0^H |z_n'(s) - z'(s)|\, ds \leq H^{1/q} \|z_n' - z'\|_{\mathcal{L}_p} \left(\frac{1}{q} + \frac{1}{p} = 1\right).$$

Hence, also from (7), it follows that $\|z_n - z\|_C \to 0$ as $n \to \infty$. The continuity of the operator $I$ is proved.

**Remark.** Consider the operator $F_y(x)$, defined by the formula

$$F_y x(t) = f[t, x(t), \tilde{x}(t - h_1(t)), \tilde{y}(t - h_2(t))]$$

$$(y \in \mathcal{L}_p(0, H),\ x \in C(0, H),\ \tilde{y}(t) = x_0'(t) \text{ for } t < 0).$$

Arguing as above, one can verify that, under the conditions of Lemma 1, this operator maps $C(0, H)$ into $\mathcal{L}_p(0, H)$ and is continuous.

Our next task is the construction of the set $T$ occurring in Theorem 1. Fix $R > 0$ (so that for $-h < t \leq 0$ the inequality $|x_0(t) - x_0| \leq R$ holds) and denote by $T_1$ the set of functions from $\mathcal{E}_0$ satisfying the inequality $\|x - x_0\|_{\mathcal{E}} \leq R$. Choose $H > 0$ so small that the inequality

$$\int_0^H m_R(t)\, dt + \|m_R(t)\|_{\mathcal{L}_p} \leq R$$

holds.

Then $IT_1 \subseteq T_1$. Indeed,

$$\|Ix - x_0\|_{\mathcal{E}} = \|Ix - x_0\|_C + \|(Ix)'\|_{\mathcal{L}_p} \leq$$

$$\leq \int_0^H |f[s, x(s), \tilde{x}(s - h_1(s)), \tilde{x}'(s - h_2(s))]|\, ds + \|m_R(t)\|_{\mathcal{L}_p} \leq R.$$

The set $T_1$ is convex, closed, and bounded; however, the operator $I$ on this set is not, generally speaking, condensing. Put $T = \overline{\text{co}}\, IT_1$. It is not hard to see that the set $T$ is also convex, closed, and bounded, and moreover $IT \subseteq T$.

**Lemma 2.** Suppose the conditions of Lemma 1 are satisfied. Suppose, in addition, that the condition

$$\text{(X)} \qquad kr^{1/p} < \begin{cases} 1, & \text{if } p > 1, \\ 1/2, & \text{if } p = 1. \end{cases}$$

Then the operator $I$ is condensing on $T$, if $H$ is sufficiently small.

**Proof.** If $\Omega \subseteq T$, then by the symbol $(\Omega)'$ we shall denote the set of derivatives of functions from the set $\Omega$. The assertion of the lemma follows from the inequalities

$$\chi_{\mathcal{E}}(I\Omega) \leq (1 + H^{1/q})\chi_{\mathcal{L}_p}[(I\Omega)'], \tag{8}$$

$$\chi_{\mathcal{L}_p}[(I\Omega)'] \leq kr^{1/p}\chi_{\mathcal{L}_p}[(\Omega)']; \tag{9}$$

$$\chi_{\mathcal{L}_p}[(\Omega)'] \leq \chi_{\mathcal{E}}(\Omega) \tag{10}$$

and condition (X).

We shall not dwell on the proof of inequalities (8), (10). Let us verify the validity of inequality (9). First note that the set $\Omega$ is compact in the space $C(0, H)$, since it is uniformly bounded and equicontinuous. Let $\{y_i;\ i = 1, 2, \dots, n\}$ be an $\varepsilon$-net of the set $(\Omega)'$ in $\mathcal{L}_p(0, H)$. Construct the sets $S_i = F_{y_i}(\Omega)$ (see the remark to Lemma 1). The sets $S_i$ are compact in $\mathcal{L}_p(0, H)$, since the operators $F_{y_i}$ are continuous. Consequently, the union of these sets

$$S = \bigcup_{i=1}^{n} S_i$$

is also compact. We show that $S$ is a $kr^{1/p}\varepsilon$-net of the set $(I\Omega)'$ in $\mathcal{L}_p(0, H)$. Let $z \in (I\Omega)'$, i.e.

$$z(t) = f[t, y(t), \tilde{y}(t - h_1(t)), \tilde{y}'(t - h_2(t))],$$

where $y \in \Omega$. Let $\|y' - y_{i_0}\|_{\mathcal{L}_p} \leq \varepsilon$. Choose in the set $S_{i_0}$ the element $u = F_{y_{i_0}}(y)$. Then

$$\|z - u\|_{\mathcal{L}_p} \leq k \left\{ \int_0^H \left|\tilde{y}'(s - h_2(s)) - \tilde{y}_{i_0}(s - h_2(s))\right|^p ds \right\}^{1/p}.$$

Arguing as in the proof of Lemma 1, we obtain

$$\left\{ \int_0^H \left|\tilde{y}'(s - h_2(s)) - \tilde{y}_{i_0}(s - h_2(s))\right|^p ds \right\}^{1/p} \leq r^{1/p}\|y' - y_{i_0}\|_{\mathcal{L}_p}.$$

Thus,

$$\|z - u\|_{\mathcal{L}_p} \leq kr^{1/p}\varepsilon,$$

i.e. from a finite $\varepsilon$-net of the set $(\Omega)'$ we can construct a compact $kr^{1/p}\varepsilon$-net of the set $(I\Omega)'$. This means precisely that inequality (9) holds. From (8)–(10) we obtain

$$\chi_{\mathcal{E}}(I\Omega) \leq \Theta\chi_{\mathcal{E}}(\Omega),$$

where $\Theta = (1 + H^{1/q})kr^{1/p}$.

It is obvious that for sufficiently small $H$, condition (X) implies the inequality $\Theta < 1$ (if $p = 1$, then $q = \infty$ and $H^{1/q} = 1$). The lemma is proved.

From Lemmas 1, 2 and the fixed-point principle for condensing operators we obtain a theorem on the solvability of problem (1)–(1′).

**Theorem 2.** Let the functions $x_0(t), h_1(t), h_2(t)$ and $f(t, x, y, z)$ satisfy conditions (I)–(X).

Then problem (1)–(1′) has a solution $x(t)$, defined on some semi-interval $(-h, H]$ $(H > 0)$.

The authors express their gratitude to Yu. G. Borisovich and to the participants of the seminar directed by him, as well as to the participants of G. A. Bezsmertnykh' s seminar, for a detailed discussion of the work and a number of useful remarks.

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Received
6 XI 1968

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