TOLMAN’ S PROBLEM IN A CENTRALLY SYMMETRIC REFERENCE SYSTEM

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Abstract

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K. P. STANYUKOVICH

TOLMAN' S PROBLEM IN A CENTRALLY SYMMETRIC REFERENCE SYSTEM

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Tolman' s problem on the motion of dust matter (when the pressure $p = 0$) is important not only as a formal example of an exact solution of the general theory of relativity, but also substantially helps in various cosmological constructions. Since Tolman' s solution is usually considered in the comoving reference system (^{1,2}), for complete information about the motion of a medium in its own gravitational field it is necessary, for some observer not connected with this reference system, to be able to calculate the relative velocity of motion of the medium. This, generally speaking, can be done, but it is more advantageous to proceed otherwise and integrate the equations written directly for a stationary central observer.

The basic equations in the central reference system and coordinates are conveniently written in the form (^{3})

$$\frac{1}{2c^2\theta^2}\left[u_r^2 - pr^2u_m^2\right] - \frac{\omega^2}{c^2}\left[(\ln v)_r + \varepsilon r^2(\ln v)_m\right] + \frac{\chi(m/r + pr^2)}{2r(1-\chi m/r)} = \frac{r^2T^0\sigma_m}{V} = \frac{\sigma_{rT}^0}{pV}; \tag{1}$$

$$-\left[(\ln V)_r - pr^2(\ln V)_m\right] + \frac{1}{2c^2\theta^2}\left[u_r^2 + \varepsilon r^2 u_m^2\right] + \frac{2}{r} + \frac{\chi(m/r - \varepsilon r^2)}{2r(1-\chi m/r)} = 0. \tag{2}$$

Here T^0 is temperature; ω is the speed of sound; $\omega^2(\partial p/\partial\varepsilon)_\sigma$; V is the specific volume; u is the 3-velocity; $\theta^2 = 1 - \frac{u^2}{c^2}$, $m = \frac{r}{\chi}(1 - e^{-\lambda}) = \int_0^r \frac{r^2}{\theta^2}\left(\varepsilon + p\frac{u^2}{c^2}\right)dr$ is a "Lagrangian" coordinate having the dimension of energy; p is pressure; σ is entropy.

Next we have the equation relating t, m, r:

$$\frac{r^2}{\theta^2}\left(\varepsilon + p\frac{u^2}{c^2}\right)\frac{\partial t}{\partial m} + \frac{\partial t}{\partial r} = 0 \tag{3}$$

and the equation determining v:

$$e^{\nu/2} = -\frac{\theta^2}{r^2 u(p+\varepsilon)\dfrac{\partial t}{\partial m}\sqrt{1-\dfrac{\chi m}{r}}}. \tag{4}$$

For dust matter, i.e., for $p = 0$, $T^0 = 0$, we have

$$\frac{u_r^2}{2c^2\theta^2} + \frac{\chi m/r}{2r(1-\chi m/r)} = \frac{r^2 T^0 \sigma_m}{V} = 0; \qquad \sigma_r = 0; \tag{5}$$

$$-(\ln V)_r + \frac{1}{2\theta^2 u^2}\left[u_r^2 + \varepsilon r^2 u_m^2\right] + \frac{2}{r} + \frac{\chi}{2r}\frac{(m/r - \varepsilon r^2)}{(1-\chi m/r)} = 0; \tag{6}$$

$$\frac{r^2}{\theta^2}\varepsilon\frac{\partial t}{\partial m} + \frac{\partial t}{\partial r} = 0; \tag{7}$$

$$e^{\nu/2} = -\frac{\theta^2}{r^2 u\varepsilon\dfrac{\partial t}{\partial m}\sqrt{1-\chi m/r}}. \tag{8}$$

Equation (5) gives

$$1 - u^2/c^2 = f(m)(1-\chi m/r). \tag{9}$$

Equation (6) takes the form

$$(\ln V)_r = \frac{2}{r} + \frac{1}{2[1-f(1-\chi m/r)]}\left\{\varepsilon r^2\left(\frac{\chi f}{r} - \frac{f_m}{f}\right) - \frac{\chi f m}{r^2}\right\}. \tag{10}$$

Since for $p = 0$, $\varepsilon v = \rho v c^2 = c^2$, we shall have

$$V_r = \frac{2V}{r} - \frac{\chi v f m}{2r^2[1-f(1-\chi m/r)]} + \frac{c^2 r^2}{2[1-f(1-\chi m/r)]}\left[\frac{\chi f}{r} - \frac{f_m}{f}\right];$$

the solution of this equation can be written in the form

$$V = r^{3/2}[(1-f)r + \chi m f]^{1/2}\left[\Phi(m) + \frac{c^2}{2}\int\frac{(\chi f/r - f_m/f)\, r^{3/2}\, dr}{[(1-f)r + \chi m f]^{3/2}}\right]. \tag{11}$$

Taking the integral, we shall have

$$V = r^{3/2}[(1-f)r + \chi m f]^{1/2}\Phi(m) - \frac{\chi r^2 c^2}{1-f}\left(f + \frac{m f_m}{1-f}\right)$$
$$- \frac{c^2 r^{3/2}[(1-f)r + \chi m f]^{1/2}}{2(1-f)^{3/2}} \times \ln \frac{\left[\sqrt{2(1-f)r + \chi m f} + \sqrt{2(1-f)r}\right]^2}{\chi m f}. \tag{12}$$

For $f = 1$, $u/c = \sqrt{\chi m/r}$, $V_r = V/r + c^2 r^2/m$, this gives

$$V = r^{3/2}\left[\Phi(m) + \frac{c^2}{2m}\int r^{1/2}\,dr\right] = \Phi(m) r^{3/2} + \frac{c^2 r^3}{3m}. \tag{13}$$

For $f = \text{const} \to 0$, $u/c = 1$

$$V = r^3\left[\frac{\Phi(m)}{r} - \frac{c^2}{2}\frac{d\ln f}{dm}\right] = r^2\Phi(m). \tag{14}$$

For a given m

$$V = \text{const}\cdot r^3. \tag{15}$$

For the subsequent analysis, it is convenient to write equation (9) in the form

$$u^2 = c^2(1-f) + \chi f m c^2/r = c^2(1-f) + 2GM_0/r, \tag{16}$$

where $M_0 = 4\pi f m/c^2$.

Let us denote $c^2(1-f) = u_0^2(M_0) = c^2(1 - M_0 c^2/4\pi m)$; then it assumes the usual "Newtonian form"

$$u^2 = u_0^2(M_0) + 2GM_0/r,$$

with

$$M_0 = (4\pi m/c^2)(1 - u_0^2/c^2). \tag{17}$$

Next, computing $\lambda = -\ln(1 - \chi m/r)$ and using equation (7), we find $t = t(m; r)$, and from equation (8) compute $\nu = \nu(m; r)$, which completely solves the problem posed—the study of the motion of a spherical "dust region" in its own gravitational field.

Let us make the limiting transition in the system of equations (5), (6), setting $u/c \ll 1$, i.e., let us pass to classical theory. As a result we obtain the system of equations

$$uu_r + GM_0/r^2 = 0, \qquad \sigma_r = 0, \tag{18}$$

where $M_0 = 4\pi m/c^2$; since $f = 1 - u_0^2/c^2 \to 1$, $\rho = 1/V$, it follows that

$$u(\ln\rho)_r + u_r + 4\pi\rho r^2 u_{M_0} + 2u/r = 0. \tag{19}$$

In this case,

$$4\pi r^2 \rho \frac{\partial t}{\partial M_0} + \frac{\partial t}{\partial r} = 0$$

$$u\frac{\partial t}{\partial M_0} = e^{-\nu/2}/4\pi\rho r^2 = -1/4\pi\rho r^2 \quad (\nu \sim u^2/c^2 \to 0).$$

The well-known solution (18) has the form (17). Thus one may draw the fundamental conclusion that the law of motion of dust matter in general relativity coincides with the law of motion in the Newtonian gravitational field. This is seen most convincingly directly from equation (5), which we shall write in the form

$$\frac{d(1-u^2/c^2)}{1-u^2/c^2} = \frac{d(1-\chi m/r)}{1-\chi m/r} = \frac{d[f(m)(1-\chi m/r)]}{f(m)(1-\chi m/r)}, \tag{20}$$

whence (9) or (17) follows.

However, the distribution of the density of the medium will be different for large and small relative velocities and when terms containing $c^2\chi$ and χ are neglected, which is obvious.

It is known that in Friedmann dust models of the Universe the law of expansion of these models is also Newtonian. This is usually explained by saying that Friedmann spaces are homogeneous and isotropic; however, Tolman models are isotropic but inhomogeneous. Consequently, the point here is not homogeneity, but only isotropy, when the effects of general relativity, such as the deflection of light rays and the motion of the perihelion of planets moving in elliptical orbits, appear only under anisotropy of the motion. Otherwise they can no longer appear. Further, one may say that the assumption $p = 0$ plainly contradicts the general-relativistic character of the equations of general relativity. We know that, since the gravitational field is itself material and must have the equation of state of an ultrarelativistic gas, $p \simeq \varepsilon$. Further, since the energy density of the gravitational field of the Metagalaxy

$$\varepsilon \approx G\overline{M}^2/a^4 \approx 10^{-7}\ \text{erg/cm}^3 \simeq \rho_{\text{M}} c^2 \tag{21}$$

corresponds to the energy density of the matter of the Metagalaxy, where $\overline{M}$ is the mass of the Metagalaxy, a is the radius of its curvature, and $\rho_{\mathrm{M}} \approx 10^{-28}$ g/cm^3 is the density of matter in the Metagalaxy, then the neglect of pressure makes the problem of describing a model of the Universe (the Metagalaxy) rather unconvincing. In this case the relativistic effects disappear and the theory automatically becomes quasi-Newtonian. In the purely Newtonian approximation this problem was solved by L. I. Sedov and M. L. Lidov ([4]).

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Note: Figure translations are in progress. See original paper for figures.

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