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Abstract

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ON THE UNIFORM APPROXIMATION OF A MONOTONE SOLUTION OF ILL-POSED PROBLEMS

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In the present paper we consider the question of solving the equation

$$A[x, z(s)] = u(x), \qquad a \leq s \leq b;\ c \leq x \leq d;\ u(x) \in U;\ z(s) \in Z, \qquad (1)$$

where $u(x)$ is given, and $z(s)$ is an unknown function, belonging respectively to the functional normed spaces U and Z; A is a continuous operator.

Suppose that for every $\bar{u}(x) \in U$ there exists, and moreover uniquely, a $z(s) \in Z$ such that $A[x, \bar{z}(s)] = \bar{u}(x)$. Suppose that the operator A^{-1} is not a bounded operator on U, i.e., problem (1) is ill-posed ([1]). The availability of some additional information about the solution makes it possible in a number of cases to make the problem well-posed in the sense of Tikhonov ([2]). Such natural information for certain physical problems is the boundedness and monotonicity of the sought function $z(s)$ ([3]).

1. Suppose an ill-posed problem (1) is given and it is known that the exact solution $\bar{z}(s)$ of the problem, corresponding to $\bar{u}(x)$, is a monotone (for definiteness, nonincreasing) function bounded by some constant B:

Introduce the set of functions $Z\downarrow$ such that $z(s) \in Z\downarrow$ if: a) $z(s)$ is a monotone nonincreasing function; b) $|z(s)| \leq B;\ a \leq s \leq b$.

Theorem 1. *The set $Z\downarrow$ is compact in the metric of the space L_α.*

Take an arbitrary sequence of functions $z_1(s), z_2(s), \dots, z_n(s), \dots$. According to Helly' s selection theorem ([4]), there exist a sequence of indices $n_0, n_1, \dots, n_k, \dots$ and a function $\bar{z}(s) \in Z\downarrow$ such that

$$\lim_{n_k \to \infty} z_{n_k}(s) = \bar{z}(s)$$

everywhere except for at most a countable number of discontinuity points of $\bar{z}(s)$. From convergence almost everywhere and the uniform boundedness of the norms in L_α there follows convergence in L_α (^{5}). The closedness of $Z\downarrow$ in L_α is evident.

2. Usually, when solving equation (1), not the exact value of the function $\bar{u}(x)$ is known, but only some approximation $\tilde{u}(x)$ to it (obtained, say, from an experiment) and an error δ such that

$$\|\tilde{u}(x) - \bar{u}(x)\|_U \le \delta.$$

Consider the set U_δ such that $\tilde{u}_\delta(x) \in U_\delta$, if

$$\|\tilde{u}_\delta(x) - \bar{u}(x)\|_U \le \delta,$$

and introduce the set of functions $\tilde{z}_\delta(s) \in Z_\delta \downarrow \in Z\downarrow$ such that

$$\|A[x, \tilde{z}_\delta(s)] - \tilde{u}_\delta(x)\|_U \le \delta^2, \qquad \tilde{u}_\delta(x) \in U_\delta. \tag{2}$$

Theorem 2. *For any $\varepsilon > 0$ there exists a $\delta_0(\varepsilon)$ such that, for any $\tilde{u}_\delta(x) \in U_\delta$,*

$$\|\tilde{z}_\delta(s) - \bar{z}(s)\|_{L_2} \le \varepsilon$$

for all $z_\delta(s) \in Z_\delta\downarrow$, if $\delta \le \delta_0(\varepsilon)$.

Lemma 1. *Let $\tilde{u}_\delta(x) \in U_\delta$, $\tilde{z}_\delta(s) \in Z_\delta\downarrow$. Then*

$$\left\|A[x, \tilde{z}_\delta(s)] - A[x, \bar{z}(s)]\right\|_U^2 \le 2\delta^2.$$

The proof is obvious:

$$\|A[x, \tilde{z}_\delta(s)] - A[x, \bar{z}(s)]\|_U^2 \le \|A[x, \tilde{z}_\delta(s)] - \tilde{u}_\delta(x)\|_U^2 + \|\tilde{u}_\delta(x) - \bar{u}(x)\|_U^2 \le 2\delta^2. \tag{3}$$

From Lemma 1 and the compactness of $Z\downarrow$, Theorem 2 follows.

3. Let $\bar{z}(s)$ be a piecewise smooth function. Take a sequence $\tilde{u}_{\delta_k}(x)$, where $\delta_k \to 0$. For each δ_k, choose some function $\tilde{z}_{\delta_k}(s) \in Z_{\delta_k}$. In this case the sequence $\tilde{z}_{\delta_k}(s)$ converges to $\bar{z}(s)$ uniformly on each segment $[\gamma, \sigma] \subset [a, b]$ that contains no discontinuity points of $\bar{z}(s)$ and no boundary points a, b.

4. For the practical determination of a function $\tilde{z}_\delta(s)$ such that

$$\|A[x, \tilde{z}_\delta(s)] - \tilde{u}_\delta(x)\|_U \le \delta, \qquad \tilde{u}_\delta(x) \in U_\delta, \tag{4}$$

it is natural to pass to grid functions. Then the problem of finding a function $\tilde{z}_\delta(s)$ satisfying relation (4) is the usual problem of quadratic programming ([6]). The only difference is that it is not necessary to seek the minimum of $\|A[x, z_\delta(s)] - \tilde{u}_\delta(x)\|$, but it is sufficient to construct a minimizing sequence until a function satisfying relation (4) is found.

This method was used by the authors in solving certain problems of astronomy ([3]).

Solving problem (4), one can find a function $\tilde{z}_\delta(s)$—an approximation to the true solution $\bar{z}(s)$—such that $\|A[x, \tilde{z}_\delta(s)] - \tilde{u}_\delta(x)\|_U \leq \delta$. To estimate the error of this approximation, introduce into the set $Z_\delta\downarrow$ the metric in which we are interested in estimating the approximation error. The approximation error will be obtained if we find

$$\sup_{z\in Z_\delta\downarrow} \rho(z(s), \tilde{z}_\delta(s)). \tag{5}$$

It is easy to see that

$$\rho(\tilde{z}_\delta(s); \bar{z}(s)) \leq \sup_{z\in Z_\delta\downarrow} \rho(z, \tilde{z}_\delta)$$

($\tilde{z}_\delta(s)$ is a certain fixed function belonging to $Z_\delta\downarrow$; $\rho(\tilde{z}_\delta, z)$ is the distance between z and $\tilde{z}_\delta$ in the metric of interest to us).

For the practical determination of the error it is necessary to introduce grids in s and x and a finite-dimensional approximation of (5). Then this problem is a quadratic programming problem ([5]).

The error estimate by this method was carried out by the authors in solving certain astronomical problems ([3]).

6. Of great interest is the question of the rate of convergence of the approximations to the true solution.

Let problem (1) be a Fredholm integral equation of the first kind

$$A[x, z(s)] = \int_a^b K(x, s) z(s)\, ds = u(x); \qquad a \leq s \leq b; \quad c \leq x \leq d, \tag{6}$$

with kernel

$$K(x, s) = C \sum_{n=1}^{\infty} \frac{\sin ns \cdot \sin nx}{n^p} + D \sum_{n=1}^{\infty} \frac{\cos ns \cdot \cos nx}{n^p}.$$

Theorem. For any $\tilde{z}_\delta(s) \in Z_\delta\downarrow$ that is an approximate solution of problem (6), the following estimate holds:

$$\|\tilde{z}_\delta(s) - \bar{z}(s)\|_{L_2} = O\left(\delta^{1/2(p+1)}\right),$$

where O depends on C, D, p, B; B is a constant bounding the solution.

In conclusion, the authors consider it their pleasant duty to express their gratitude to Acad. A. N. Tikhonov for posing the problem and supervising the work, to V. B. Glasko for supervising the work, and also to Sh. A. Alimov for numerous discussions.

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Note: Figure translations are in progress. See original paper for figures.