# ON AN EXAMPLE OF A GRAPH HAVING NO TRANSITIVE AUTOMORPHISM GROUP

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## Abstract

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# ON AN EXAMPLE OF A GRAPH HAVING NO TRANSITIVE AUTOMORPHISM GROUP

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1. In paper ([1]) a construction was indicated which makes it possible, for a given graph $\Gamma$, to construct in an invariant way the matrix algebra $\mathfrak{A}(\Gamma)$. In the present paper we consider graphs to which there corresponds a special class of algebras—cells.

An a priori definition of a cell:

A matrix algebra $\mathfrak{A}$ is called a **cell** if it has the following properties:

1. $\mathfrak{A}$ is invariant under transposition.
2. In $\mathfrak{A}$ there exists a basis $e_0 = E, e_1, \dots, e_k$ such that $e_i$ is a matrix consisting of zeros and ones and having $n_i$ ones in each row and each column,

$$n = \sum_{i=1}^{k} n_i + 1; \qquad \sum_{i=}^{k} e_i$$

is a matrix all of whose entries are equal to 1.

Let $\Gamma$ be a graph and $\mathfrak{A}(\Gamma)$ its algebra. The condition that $\mathfrak{A}(\Gamma)$ is a cell has the following geometric meaning. To each ordered sequence $(g_0, g_1, \dots, g_m)$ of vertices of the graph $\Gamma$ $(m \geqslant 1)$ there corresponds the sequence $(\xi_1, \dots, \xi_m)$, where $\xi_i = 1$ if $(g_{i-1}, g_i)$ is an edge of $\Gamma$, and $\xi_i = 0$ otherwise. If $\mathfrak{A}(\Gamma)$ is a cell, then for any fixed sequence $\Xi_0 = (\xi_1, \dots, \xi_k)$ of zeros and ones the number of sequences of vertices $(g \dots)$ and $(h \dots)$ to which the sequence $\Xi_0$ corresponds is the same for any two vertices $g$ and $h$.

II. In ([1]) the following was stated.

**Hypothesis.** The automorphism group $\operatorname{Aut}\Gamma$ of a graph $\Gamma$, the algebra $\mathfrak{A}(\Gamma)$ of which is a cell, is transitive on its vertices.

Below a counterexample to this hypothesis is constructed.

A. Let $\Gamma$ be a graph, $G = \operatorname{Aut}\Gamma$, and $\mathfrak{Z}(G)$ the algebra of matrices commuting with the group $G$. $\mathfrak{Z}(G)$ has a basis $f_i$, $i \in \mathfrak{I}$, all of whose matrices consist of 0 and 1. It is known that $\mathfrak{A}(\Gamma)$ is a subalgebra of $\mathfrak{Z}(G)$, and

$$e_i = \sum_{j \in \mathfrak{I}_i} f_j,$$

where the $\mathfrak{I}_i$ are suitable disjoint subsets of the set $\mathfrak{I}$.

B. If $n = 2p$, $p$ is prime, and $G$ is a primitive group, transitive but not doubly transitive on the vertices of the graph $\Gamma$ (with $n$ vertices), then, as shown in ([2]), $\dim \mathfrak{Z}(G) = 3$, $2p = m^2 + 1$, $n_1 = m(m+1)/2$, $n_2 = m(m-1)/2$. Hence, and from what was said above, it follows that $\dim \mathfrak{A}(\Gamma) \leqslant 3$; the case $\dim \mathfrak{A}(\Gamma) = 2$ is trivial and corresponds to the complete graph $\Gamma$. Therefore we shall assume that $\mathfrak{A}(\Gamma) = \mathfrak{Z}(G)$. Note that the matrices $e_i$ in this case are symmetric and are, therefore, adjacency matrices of undirected graphs without multiple edges. Moreover, $\mathfrak{A}(e_i) = \mathfrak{Z}(G)$, $i > 0$.

As was indicated in ([2]), and also in a later paper ([3]), it is unknown whether there exists a primitive not doubly transitive group $G$ for $p > 5$.

C. For $p = 13$ all three-dimensional cells were constructed whose automorphism group contains the element $\sigma = (1, 2, \dots, p)(p+1, \dots, 2p)$.

It turned out that the automorphism groups of all the corresponding graphs are intransitive. Hence, and from the results of [2], it follows that

*Every primitive transitive permutation group of degree 26 is doubly transitive.*

III. In view of IIA, the basic matrix of the $e_1$-cell, permutable with $\sigma$, has the form

$$\begin{pmatrix} A_1 A_2 \\ A_3 A_4 \end{pmatrix},$$

where $A_3' = A_2$, $A_1' = A_1$, $A_4' = A_4$, $A_i = \sum_{j \in \mathfrak{S}_i} \tau^j$, where $\tau$ is the permutation matrix of $(0, 1, \dots, p-1)$, and $\mathfrak{S}_i$ are suitable subsets of residues $\operatorname{mod} p$; $\mathfrak{S}_1$ and $\mathfrak{S}_4$ do not contain 0.

The choice of the suitable sets $\mathfrak{S}_i$ was carried out by direct search with the aid of a computer. In all, 104 different collections of $\mathfrak{S}_i$ were obtained. In all cases either

$$\mathfrak{S}_1 = \{1, 3, 4, 9, 10, 12\}, \quad \mathfrak{S}_4 = \{2, 5, 6, 7, 8, 11\},$$

or conversely; for fixed $\mathfrak{S}_2$ these two cases are, evidently, isomorphic. For given $\mathfrak{S}_1$ and $\mathfrak{S}_4$, 52 collections of $\mathfrak{S}_2$ were obtained; the corresponding graphs are pairwise isomorphic. The isomorphisms consist of permutations belonging to the group generated by the substitutions

$$\begin{pmatrix} E & 0 \\ 0 & \tau \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \rho \\ \rho^{-1} & 0 \end{pmatrix}$$

for

$$\rho = (0)(1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7);$$

one of the admissible collections is

$$\mathfrak{S}_2 = \{0, 1, 3, 9\}.$$

The intransitivity of the obtained graph $\Gamma$ is established by comparing the subgraphs $\Gamma_1$ and $\Gamma_2$, spanned by the vertices adjacent respectively to the 1st and 14th vertices of the graph $\Gamma$ (Fig. 1).

**Fig. 1**

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## REFERENCES

1. B. Yu. Weisfeiler, L. L. Leman, *Scientific and Technical Information*, No. 9, issue 2 (1968).

2. H. Wielandt, *Math. Zs.*, **63**, 478 (1955).

3. D. G. Higman, *Math. Zs.*, **86**, No. 2, 145 (1964).

*Note: Figure translations are in progress. See original paper for figures.*