ON A PROJECTION OPERATOR OF S. L. SOBOLEV TYPE

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 513.88

MATHEMATICS

V. R. PORTNOV

ON A PROJECTION OPERATOR OF S. L. SOBOLEV TYPE

(Presented by Academician S. L. Sobolev on 9 IV 1969)

In the work of S. L. Sobolev (⌒1), for the space $L_p^{(m)}(\Omega)$ a projection operator Π was constructed whose range is the space of polynomials of degree not greater than $m-1$. This operator Π has the property that for any function $u(x) \in L_p^{(m)}(\Omega)$ there is a representation

$$u(x) - \Pi u(x) = \sum_{|\alpha|=m} Q_\alpha(D^\alpha u(x)), \tag{1}$$

where Q_α is a certain linear bounded operator mapping $L_p(\Omega)$ into itself. Representations of the form (1) were later generalized in various directions (see ([2]-4)). In (⌒5), applications of such representations were given to the proof of solvability of the first boundary-value problem for one class of degenerate equations.

In the present note a projection operator Π is constructed, and a representation of the form (1) is obtained for a certain collection $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_R$ of partial differential operators that are products of ordinary differential operators with respect to different variables. The corresponding result of (⌒5) is contained here as a special case.

Let us first introduce a number of notations: n and R are a fixed pair of natural numbers; the letters i and k below will denote natural numbers not exceeding n and R, respectively; $\bar{E}$ is the extended real line; E_n is the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$. The remaining notations will be introduced in the course of the exposition.

Suppose that for each i an interval (a_i, c_i) is given on the axis Ox_i of the space E_n, where a_i and $c_i \in \bar{E}$. This means that $-\infty \le a_i < c_i \le \infty$. Put $D = (a_1, c_1) \times \dots \times (a_n, c_n)$.

Let, further, to each k there correspond a set $(m_{1,k}, m_{2,k}, \dots, m_{n,k})$ of n non-negative integers and an operator $\mathcal{L}_k = \mathcal{L}_{1,k}\mathcal{L}_{2,k} \dots \mathcal{L}_{n,k}$, whose domain is the space $C^{(\infty)}(D)$, and for each i

$$\mathcal{L}_{i,k} = \begin{cases} \dfrac{\partial^{m_{i,k}}}{\partial x_i^{m_{i,k}}} + \displaystyle\sum_{j=1}^{m_{i,k}} b_{i,k}^{(j)}(x_i) \dfrac{\partial^{j-1}}{\partial x_i^{j-1}}, & \text{if } m_{i,k} > 0, \\ I \ (I \text{ is the identity operator}), & \text{if } m_{i,k} = 0, \end{cases}$$

where $b_{i,k}^{(j)}(x_i) \in L_1^{\mathrm{loc}}(D)$ for all $j = 1, \dots, m_{i,k}$.

If $m_{i,k} > 0$, then the operator $\mathcal{L}_{i,k}$ may be regarded as an ordinary differential operator on the interval (a_i, c_i). Let $\{\varepsilon_{i,k}^{(j)}(x_i)\}$ $(j = 1, \dots, m_{i,k})$ be its fundamental system of solutions on the indicated interval. We denote the linear span of this system by $S_{i,k}$. If $m_{i,k} = 0$, then put $S_{i,k} = \{0\}$, where $\{0\}$ denotes the set consisting of the single function identically equal to zero on (a_i, c_i). This notation will be used below.

For each pair (i, k) let us consider the set $\Psi_{i,k}$ of all possible tuples $\tau = (k_1, \dots, k_l)$ $(1 \le l \le R$ and l, generally speaking, depends on $\tau)$ of natural numbers satisfying the following conditions: 1) $k < k_1 < \dots < k_l \le R$; 2) there is a tuple $(i_1, i_2, \dots, i_R)$ of R natural numbers not exceeding n such that $m_{i_1,1} > 0, \dots, m_{i_R,R} > 0$, and, moreover,

$$i_k = i_{k_1} = \dots = i_{k_l} = i,$$

while $i_j \ne i$ for $j > k$ and $j \ne k_1, \dots, j \ne k_l$.

For some pairs (i, k) it may happen that $\Psi_{i,k} = \emptyset$. For example, $\Psi_{i,R} = \emptyset$ for all i, and $\Psi_{i,k} = \emptyset$ when $m_{i,k} = 0$.

If $\Psi_{i,k} \ne \emptyset$, then for any tuple $\tau = (k_1, \dots, k_l) \in \Psi_{i,k}$ we introduce the sets $S_{i,k}^{(\tau)}$ and $\Gamma_{i,k}^{(\tau)}$ of functions defined on the interval (a_i, c_i):

$$S_{i,k}^{(\tau)} = \bigcap_{1 \le j \le l} S_{i,k_j}; \qquad \Gamma_{i,k}^{(\tau)} = S_{i,k} \cap S_{i,k}^{(\tau)}.$$

Now, for each pair (i, k), we formulate

Condition A. One of the following assertions is true:
1) $\Psi_{i,k} = \emptyset$;
2) $\Psi_{i,k} \ne \emptyset$, and for every $\tau \in \Psi_{i,k}$ there exists a subspace

$$\nabla_{i,k}^{(\tau)} \subset S_{i,k}^{(\tau)}$$

such that

$$S_{i,k}^{(\tau)} = \Gamma_{i,k}^{(\tau)} \oplus \nabla_{i,k}^{(\tau)},$$

and

$$S_{i,k} \cap \nabla_{i,k} = \{0\},$$

where $\nabla_{i,k}$ is the space spanned by

$$\bigcup_{\tau\in\Psi_{i,k}} \nabla_{i,k}^{(\tau)}.$$

(For the notation and terminology, see ([6]), pp. 30 and 44, 46.)

Everywhere in what follows it is assumed that for every pair (i,k) condition A is fulfilled and, in the case when $\Psi_{i,k}\neq\emptyset$, the family of spaces

$$\{\nabla_{i,k}^{(\tau)}\} \quad (\tau\in\Psi_{i,k})$$

is fixed.

Lemma. Let $m_{i,k}>0$. Then there exists a system of functions

$$\{f_{i,k}^{(j)}(x_i)\} \quad (j=1,\dots,m_{i,k}),$$

defined on the interval (a_i,c_i), finite and infinitely differentiable, which has the property that

$$\int_{a_i}^{c_i} f_{i,k}^{(j)}(x_i)e_{i,k}^{(s)}(x_i)\,dx_i=\delta_{js},$$

where δ_{js} is the Kronecker symbol, and, in the case when $\Psi_{i,k}\neq\emptyset$,

$$\int_{a_i}^{c_i} f_{i,k}^{(j)}(x_i)v(x_i)\,dx_i=0$$

for any function $v(x_i)\in\nabla_{i,k}$ and for all $j=1,\dots,m_{i,k}$.

If $m_{i,k}>0$, then the functions $\{e_{i,k}^{(j)}(x_i)\}$ and the functions $\{f_{i,k}^{(j)}(x_i)\}$ $(j=1,\dots,m_{i,k})$ appearing in the lemma will be regarded as fixed and defined on the whole domain D.

We construct the projection operator $\Pi_{i,k}$, defining it on functions $u(x)$ from $L_1^{\mathrm{loc}}(D)$, in the case $m_{i,k}>0$, as follows:

$$\Pi_{i,k}u(x)=\sum_{j=1}^{m_{i,k}}\left(\int_{a_i}^{c_i} f_{i,k}^{(j)}(\xi_i)\,u(x_1,\dots,x_{i-1},\xi_i,x_{i+1},\dots,x_n)\,d\xi_i\right)e_{i,k}^{(j)}(x_i).$$

If $m_{i,k}=0$, then $\Pi_{i,k}$ will be regarded as the zero operator on the space $L_1^{\mathrm{loc}}(D)$.

We now define the operators Π_k and Π, mapping the space $L_1^{\mathrm{loc}}(D)$ into itself. Put

$$\Pi_k=I-(I-\Pi_{1,k})(I-\Pi_{2,k})\cdots(I-\Pi_{n,k}) \qquad (k=1,\dots,R),$$

$$\Pi=\Pi_1\Pi_2\cdots\Pi_R \qquad (I \text{ is the identity operator}).$$

Theorem 1. The following equalities hold:

$$\Pi_k^2 = \Pi_k, \qquad \Pi_k \Pi = \Pi \quad (k = 1, \dots, R); \qquad \Pi^2 = \Pi.$$

Definition. An additive and homogeneous operator Q mapping the space $L_1^{\text{loc}}(D)$ into itself will be called **strongly continuous** if for every compact set $K^{(1)} \subset D$ there exists a compact set $K^{(2)} \subset D$, depending, generally speaking, on $K^{(1)}$, such that for all functions $w(x) \in L_1^{\text{loc}}(D)$ the inequality

$$\|Qw(x)\|_{L_1(K^{(1)})} \leq C \|w(x)\|_{L_1(K^{(2)})},$$

holds, where C is a real number independent of $w(x)$.

Theorem 2. For any function $u(x) \in C^{(\infty)}(D)$ and any k, the equality

$$u(x) - \Pi_k u(x) = T_k(\mathcal{L}_k u(x)), \tag{2}$$

holds, where T_k is a strongly continuous operator in the space $L_1^{\text{loc}}(D)$, possessing the property that $T_k w(x) = 0$ implies $w(x) = 0$.

Theorem 3. For any function $u(x) \in C^{(\infty)}(D)$ and any collection $\beta = (\beta_1, \dots, \beta_n)$ of nonnegative integers such that $\beta_i \leq \min_{1 \leq k \leq R} m_{i,k}$ $(i = 1, \dots, n)$, the equality

$$\mathcal{D}^\beta (u(x) - \Pi u(x)) = \sum_{k=1}^{R} Q_k^{(\beta)} (\mathcal{L}_k u(x)), \tag{3}$$

holds, where $Q_k^{(\beta)}$ $(k = 1, \dots, R)$ are strongly continuous operators in the space $L_1^{\text{loc}}(D)$.

Remark 1. The operators $\Pi_{i,k}$, Π_k, and Π are, obviously, strongly continuous in the space $L_1^{\text{loc}}(D)$.

Remark 2. Equalities (2) and (3) are in fact valid, of course, for a class of functions broader than $C^{(\infty)}(D)$. This class will not be defined here; however, let us note that, for example, equality (3) with $\beta = (0, \dots, 0)$, and in the case when all the operators $\mathcal{L}_{i,k}$ with $m_{i,k} > 0$ have constant coefficients, is valid for all functions for which in the domain D there exist generalized, in the sense of S. L. Sobolev, operators $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_R$, as is easily verified by means of passage to the limit in equality (3).

Received
24 III 1969

REFERENCES

1. S. L. Sobolev, *Some applications of functional analysis in mathematical physics*, Novosibirsk, 1962.

2. O. V. Besov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **89**, 5 (1967).

3. O. V. Besov, Matem. sborn., **73** (115), issue 3, No. 4, 599 (1967).

4. V. P. Il' in, Sibirsk. matem. zhurn., **8**, No. 3, 573 (1967).

5. V. R. Portnov, *Embedding theorems and their applications*, Baku, May, 1966; Moscow, 1968.

6. P. Halmos, *Finite-dimensional vector spaces*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.