



Soviet-era science, translated into English

APPROXIMATION OF THE THEORY OF DYADIC BICOMPACTS

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.45132>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.83

MATHEMATICS

A. V. ARKHANGELSKII

APPROXIMATION OF THE THEORY OF DYADIC BICOMPACTS

(Presented by Academician P. S. Aleksandrov on 24 V 1968)

§ 1. **Preliminary.** In the article, on the basis of two new concepts and their interweaving, a unified approach is given to the proof of many theorems in the theory of dyadic bicomacts. A fundamental role here is played by the well-known theorem of Hewitt–Marczewski–Pondiczery⁽¹⁻³⁾. All spaces are assumed to be Hausdorff. By $s(X)$ we denote the density of a space, by $w(X)$ the weight of a space, and by $|X|$ the cardinality of a set X .

Let τ be an infinite cardinal number, X a topological space, and X' its subspace.

0.1. We shall call X' τ -monolithic in X if for every $A \subset X'$ such that $|A| \leq \tau$, $[A]_X$ is a bicomact of weight $\leq \tau$.

0.2. We shall say that X τ -suppresses X' if from $\lambda \geq \tau$ and $A \subset X'$, $|A| \leq 2^\lambda$, it follows that there exists $\tilde{A} \subset X$ for which $[\tilde{A}] \supset A$ and $|\tilde{A}| \leq \lambda$.

Let $X = \Pi\{X_\alpha : \alpha \in M\}$, and in each X_α choose a point x_α^* . By $X(\tau)$ we denote the subspace $\{x \in X : |\{\alpha \in M : x_\alpha \neq x_\alpha^*\}| \leq \tau\}$ of the space X . Suppose further that X_α is a subspace of the space Y_α , and that Y_α τ -suppresses X_α for every $\alpha \in M$.

Put $Y = \Pi\{Y_\alpha : \alpha \in M\}$. Then

1) *If every X_α is τ -monolithic in itself (in Y_α), then $X(\tau)$ is also τ -monolithic in itself (in Y).*

2) *The space Y τ -suppresses its subspace $X(\tau)$.*

3) *If $[X_\alpha] = Y_\alpha$ for every $\alpha \in M$, then $[X(\tau)] = Y$.*

We prove 1). Let $\pi_\alpha : X \rightarrow X_\alpha$ be the projection onto the factor. Consider $C \subset X(\tau)$ for which $|C| \leq \tau$, and $C_\alpha = \pi_\alpha(C)$, $\alpha \in M$. Put $M_C = \{\alpha \in M : C_\alpha \neq \{x_\alpha^*\}\}$. Then $|M_C| \leq \tau$. Consider $F_\alpha = [C_\alpha]_{X_\alpha}$. From $|C_\alpha| \leq |C| \leq \tau$ it follows that F_α , for every $\alpha \in M$, is a bicomact of weight $\leq \tau$. Moreover, if $\alpha \notin M_C$, then $F_\alpha = \{x_\alpha^*\}$. Hence $F = \Pi\{F_\alpha : \alpha \in M\}$ is a bicomact, and F is homeomorphic to the space $\Pi\{F_\alpha : \alpha \in M_C\}$, while the weight of the latter is

not greater than τ , since $|M_C| \leq \tau$. We have $[C]_{X(\tau)} \subset F \subset X(\tau)$, i.e. $[C]_{X(\tau)}$ is a bicomact of weight $\leq \tau$.

We prove 2). Let $\lambda \geq \tau$ and $A \subset X(\tau)$, $|A| \leq 2^\lambda$. Put $A_\alpha = \pi_\alpha(A)$ and $M_A = \{\alpha \in M : A_\alpha \neq \{x_\alpha^*\}\}$. Clearly, $|M_A| \leq 2^\lambda$ and $|A_\alpha| \leq 2^\lambda$ for every $\alpha \in M$. Our assumptions allow us, for all α , to choose in Y_α a set \tilde{A}_α for which $|\tilde{A}_\alpha| \leq \lambda$ and $[\tilde{A}_\alpha]_{Y_\alpha} \supset A_\alpha$. Moreover, if $\alpha \notin M_A$, then as \tilde{A}_α one may take the set $\{x_\alpha^*\}$. Put $A^* = \Pi\{\tilde{A}_\alpha : \alpha \in M\}$. Then $[A^*] \supset A$ and A^* is homeomorphic to the space $\Pi\{\tilde{A}_\alpha : \alpha \in M_A\}$. But the density of the latter, by the Hewitt–Marczewski–Pondiczery theorem, does not exceed λ . Hence there exists $\tilde{A} \subset A^* \subset Y$ for which $|\tilde{A}| \leq \lambda$ and $[\tilde{A}] = [A^*] \supset A$.

3) is obvious.

Let $f : X \rightarrow Y$ be a continuous mapping, $X' \subset X$, $Y' = fX'$. Then the following assertions hold:

4) *If X' is τ -monolithic in itself (in X), then Y' is τ -monolithic in itself (in Y).*

5) *If X τ -suppresses X' , then Y τ -suppresses Y' .*

Proof. Let $B \subset Y'$. Choose $x(y) \in f^{-1}(y)$ for each $y \in Y'$. Put $A = \{x(y) : y \in B\}$. Then $|A| = |B|$ and $fA = B$.

If $|B| \leq \tau$, then also $|A| \leq \tau$; therefore $[A]_{X'}$ is a bicomactum of weight $\leq \tau$. Then

$f[A]_{X'} = [B]_{Y'}$, and by a well-known theorem of P. S. Aleksandrov, $w([B]_{Y'}) \leq w([A]_{X'}) \leq \tau$. This proves 4). If $|B| \leq 2^\lambda$, where $\lambda \geq \tau$, then $|A| \leq 2^\lambda$, and hence there exists $\tilde{A} \subset X$ for which $|\tilde{A}| \leq \lambda$ and $[\tilde{A}] \supset A$. Put $\tilde{B} = f\tilde{A}$. From the continuity of f it follows that $[\tilde{B}] \supset f[\tilde{A}] \supset fA = B$; obviously, $|\tilde{B}| \leq |\tilde{A}| \leq \lambda$. 5) is proved.

§ 2. **Definition 1.** A topological space X is called **dense** if for each cardinal number τ in X there exists everywhere dense τ -monolithic subspace.

Definition 2. A topological space X is called **adherent** if for each infinite cardinal number τ in X there exists everywhere dense subspace τ -subdued by X .

Definition 3. A space X is called a **dant space** if for each infinite cardinal number τ there exists an everywhere dense subspace X' in X which simultaneously:

- a) is τ -monolithic in itself;
- b) is τ -subdued by the space X .

Remarks. Every bicomactum with a countable base is a dant space. Every space with a countable base is adherent. The sum of a countable set of adherent spaces is, obviously, an adherent space.

Theorem 1. *If X τ -subdues itself, then $s(X) \leq \tau$.*

Proof. Let $A \subset X$ and $|A| \leq 2^{2^\tau}$. We shall show that then there exists $B \subset X$ for which $|B| \leq \tau$ and $[B] \supset A$. From $2^{2^\tau} \geq \tau$, $2^\tau \geq \tau$, and $\tau \geq \tau$, in view of the hypothesis of the theorem and .2) it follows that there exist $B_1 \subset X$, $B_2 \subset X$, and $B \subset X$ for which $[B_1] \supset A$ and $|B_1| \leq 2^\tau$, $[B_2] \supset B_1$ and $|B_2| \leq 2^\tau$, $[B] \supset B_2$ and $|B| \leq \tau$. Then $[B] \supset A$. Since X is a Hausdorff space, for any $C \subset X$ and $D \subset X$ from $[C] \supset D$ it follows that $|D| \leq 2^{2^{|C|}}$.

Hence, if $A \subset X$ and $|A| \leq 2^{2^\tau}$, then $|A| \leq 2^{2^\tau}$. Consequently, $|X| < 2^{2^\tau}$. Taking X for A , on the basis of what was said at the beginning of the proof we conclude that in X there exists an everywhere dense set B of cardinality not exceeding τ . The theorem is proved.

Theorem 2. *A continuous image of a dant (adherent, dense) space is a dant (respectively adherent, dense) space (see 4) and 5)).*

Theorem 3. *The product of any set of dant (adherent, dense) spaces is a dant (respectively adherent, dense) space (see 1), 2) and 3)).*

From the remarks made above and Theorem 1 it follows that

Theorem 4. *Every dyadic bicom pactum is a dant space.*

Theorem 5. *Every dense (in particular, dant) space X is a bicom pactum.*

We use Definition 1 in the case $\tau = |X|$.

Definition 4 (see (5)). The **weak tightness** $t_c(X)$ of a topological space X is the least cardinal number τ such that, if $M \subset X$ and $M \neq [M]$, then there exist a point $x \in X \setminus M$ and a set $M' \subset M$ for which $|M'| \leq \tau$ and $x \in [M']$.

Theorem 6. *For every dant space X , $t_c(X) = w(X)$.*

Proof. Obviously, $t_c(X) \leq w(X)$. Put $\tau_0 = t_c(X)$ and consider $X' \subseteq X$, which is τ_0 -monolithic in itself, is τ_0 -subdued by the space X , and is everywhere dense in X . If $X' \neq X$, then there exist $A \subset X'$ and $x_0 \in X \setminus X'$ such that $|A| \leq \tau_0$ and $[A] \ni x_0$.

But $[A]_{X'}$ is a bicom pactum, and therefore $X' \supset [A]_{X'} = [A]_X \ni x_0$. Hence $X = [X'] = X'$, i.e. X τ_0 -subdues itself and X is τ_0 -monolithic in itself. From the first, by Theorem 1, it follows that $s(X) \leq \tau_0$. From this and the second we conclude that $w(X) \leq \tau_0 = t_c(X)$. Hence, $t_c(X) = w(X)$.

Corollary 1. For dyadic bicom pacta, weak tightness coincides with weight.

Corollary 2 (A. S. Esenin-Vol' pin). For every dyadic bicom pactum X , $w(X) = \sup\{\chi(x, X) : x \in X\}$ (see (8)).

Let me recall a concept due to V. I. Ponomarev. The external π -weight of X_0 in a space $X \supset X_0$ is the minimum of the cardinalities of external π -bases of X_0 in X , i.e., of such families γ of nonempty open subsets of X that, if U is

open in X and $U \cap X_0 \neq \Lambda$, then there exists $V \in \gamma$ contained in U . Notation: $\pi(X_0, X)$.

Theorem 7. If P is a closed subspace of a dyadic space X , then $w(P) \leq \pi(P, X)$.

Proof. Let γ be an external π -base of P in X and let $|\gamma| = \pi(P, X)$. Put $\tau_0 = |\gamma|$. Consider $X_0 \subset X$, τ_0 -monolithic in itself and everywhere dense in X . For each $U \in \gamma$ fix $x(U) \in X_0 \cap U$ and put $A = \{x(U) : U \in \gamma\}$. Then $|A| \leq |\gamma| = \tau_0$ and $A \subset X_0$. Therefore $[A]_X = [A]_{X_0}$ is a bicompactum of weight $\leq \tau$. But, as follows easily from the construction, $P \subset [A]_X$, whence $w(P) \leq \tau_0$.

Corollary 1. The weight of a dyadic (in particular, Cantor) space is equal to its π -weight.

Corollary 2. Under an irreducible continuous mapping of a dyadic (in particular, Cantor) space, weight is preserved (see ⁽⁶⁾—the analogue for dyadic bicompacta).

Corollary 3. If X is an open subspace of a dyadic space X and $w(X_0) \leq \tau$, then also $w([X_0]_X) \leq \tau$.

Corollary 4. If X is dyadic and $X_0 \subset X$, $[X_0] = X$, then $w(X) = w(X_0)$.

Corollary 5. Among bicompact extensions of spaces with a countable base, the metrizable ones, and only they, are dyadic spaces.

Theorem 8. If ξ is a family of nonempty open subsets of a flattened (in particular, Cantor) space X , and $|\xi| = \tau$, where τ is a regular attainable* cardinal number, then there exists $\xi' \subseteq \xi$ such that $|\xi'| = |\xi|$ and $\bigcap \{G : G \in \xi'\} \neq \Lambda$.

Proof. There exists $\tau_1 < \tau$ for which $\tau \leq 2^{\tau_1}$. In X there is an everywhere dense subspace X_1 , τ_1 -suppressed in X . Choose $x(G) \in G \cap X_1$ for each $G \in \xi$ and put $A = \{x(G) : G \in \xi\}$. Then $|A| \leq \tau \leq 2^{\tau_1}$ and $A \in X_1$. There therefore exists $\bar{A} \subset X$ such that $|\bar{A}| \leq \tau_1 < \tau$ and $[\bar{A}] \supset A$. Since the G -neighborhood of the point $x(G) \in A$, the last relation permits, for each $G \in \xi$, the choice of $y(G) \in \bar{A} \cap G$. The set $B = \{y(G) : G \in \xi\}$ is contained in \bar{A} ; hence $|B| \leq \tau_1 < \tau$. But $G \cap B \neq \Lambda$ for each $G \in \xi$. Put $\xi_y = \{G \in \xi : G \ni y\}$, $y \in B$. Then $\xi = \bigcup \{\xi_y : y \in B\}$, and since τ is a regular number, $|\xi_{y_0}| = |\xi|$ for some $y_0 \in B$. The family ξ_{y_0} is the desired one.

Corollary 1. If a decreasing transfinite sequence of nonempty open subsets of a flattened (in particular, Cantor) space is not cofinal with the order type of any regular unattainable cardinal number, then the intersection of its elements is nonempty.

Corollary 2. If ξ is a family of open subsets of a flattened (in particular, Cantor) space X and $|\xi|_x \leq \tau$ for all $x \in X$ (where $|\xi|_x = |\{U \in \xi : U \ni x\}|$), then $|\xi| \leq \tau$.

Corollary 3. If the bicomact extension bX of a metric space X is a Cantor space, then X and bX have a countable base.

It is not known whether the statement analogous to the last is true for every X satisfying the first axiom of countability (see (7)).

Theorem 9. A dyadic subspace X of a regular extremally disconnected space is finite.

* Attainability of a number τ means that there exists $\tau_1 < \tau$ for which $\tau \leq 2^{\tau_1}$. As is known, one may assume that unattainable regular cardinal numbers do not exist (see (4), pp. 165-166).

In fact, in an infinitely dense space there is a nontrivial convergent sequence.

Let τ be an infinite cardinal number, X a space, X_0 its subspace, $A \subset X$ any subset, and $\mathcal{F}(X)$ the space of all closed subsets of X with the Vietoris topology. Put

$$\mathcal{F}_\tau(X_0) = \{F \in \mathcal{F}(X) : F \subset X_0 \text{ and } wF \leq \tau\},$$

$$\mathcal{H}(A) = \{F \in \mathcal{F}(X) : F \subset A \text{ and the set } F \text{ is finite}\}.$$

By $S(A)$ we denote some everywhere dense subset of A of least cardinality. Then:

U6). If $[X_0] = X$, then $[\mathcal{F}_\tau(X_0)] = \mathcal{F}(X)$ (obviously).

U7). If X τ -suppresses X_0 , then $\mathcal{F}(X)$ τ -suppresses $\mathcal{F}_\tau(X_0)$.

Proof. Let $\lambda \geq \tau$, $\mathcal{P} \subset \mathcal{F}_\tau(X_0)$, and $|\mathcal{P}| \leq 2^\lambda$. Put

$$P = \bigcup \{S(F) : F \in \mathcal{P}\}.$$

For each $F \in \mathcal{P}$, $|S(F)| \leq w(F) \leq \tau$ and $F \subset X_0$; hence $|P| \leq 2^\lambda$ and $P \subset X_0$. Since X τ -suppresses X_0 , there exists $Q \subset X$ such that $|Q| \leq \lambda$ and $[Q]_X \supset P$. Then $[\mathcal{H}(Q)] \supset \mathcal{P}$. Obviously, $|\mathcal{H}(Q)| \leq |Q| \leq \lambda$ —U7) is proved.

U8). If X_0 is τ -monolithic, then $\mathcal{F}_\tau(X_0)$ is also τ -monolithic.

Proof. Let $\mathcal{M} \subset \mathcal{F}_\tau(X_0)$ and $|\mathcal{M}| \leq \tau$. Put

$$M = \bigcup \{S(F) : F \in \mathcal{M}\}.$$

Then $|M| \leq \tau$ and $M \subset X_0$. Therefore $[M]_{X_0}$ is a compactum of weight $\leq \tau$, and if $F \in \mathcal{M}$, then $F \subset [M]_{X_0}$. Hence $\{F \in \mathcal{F}(X) : F \subset [M]_{X_0}\}$ is a compactum of weight $\leq \tau$, lying in $\mathcal{F}_\tau(X_0)$ and containing the set \mathcal{M} . This implies U8).

Theorem 10. If X is a Dantov space (dense, tightly adjacent), then $\mathcal{F}(X)$ is also a Dantov space (dense, tightly adjacent) (see U6), U7), and U8)).

§ 3. **Theorem 11.** The preimage of a tightly adjacent space under an open s -mapping is tightly adjacent.

Theorem 12. The preimage of a Dantov space under an open-closed finite-to-one mapping is a Dantov space.

It is not known whether analogues of Theorems 10, 11, and 12 are true for dyadic compacta.

Theorem 13. A Dantov space X that is a continuous image of an ordered compactum has a countable base.

The last result contains, by virtue of Theorem 4, the known assertion of Mardešić and Papić on dyadic compacta ⁽⁹⁾.

I note that at present there is no example of a Dantov space that is not a dyadic compactum.

Mechanics and Mathematics Faculty
M. V. Lomonosov Moscow State University

Received
24 IV 1968

REFERENCES

1. E. Hewitt, Bull. Am. Math. Soc., **52**, 641 (1946).
2. E. S. Pondiczery, Duke Math. J., **11**, 835 (1944).
3. E. Marczewski, Fund. Math., **34**, 427 (1947).
4. I. L. Gillman, M. Jerison, *Rings of Continuous Functions*, Princeton, 1960.
5. A. V. Arkhangel'skii, V. I. Ponomarev, DAN, **182**, No. 5 (1968).
6. R. Engelking, A. Pelczyński, Coll. Math., **11**, No. 1, 55 (1963).
7. B. A. Efimov, Tr. Mosk. matem. obshch., **14**, 211 (1965).
8. A. S. Esenin-Volpin, DAN, **68**, 441 (1949).
9. S. Mardešić, P. Papić, Glasnik Math., Fiz. i Astr., ser. II, **17**, 3 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.