TRIANGULAR REPRESENTATIONS OF DISSIPATIVE OPERATORS WITH RESOLVENT OF EXPONENTIAL TYPE

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Abstract

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TRIANGULAR REPRESENTATIONS OF DISSIPATIVE OPERATORS WITH RESOLVENT OF EXPONENTIAL TYPE

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A bounded linear operator A, acting in a separable Hilbert space $\mathfrak{H}$, will be assigned to the class $\Lambda^{(\exp)}$ if:
1) the spectrum of the operator A is concentrated at zero;
2) $A_I = (A - A^*)/2i \geq 0$;
3) $(I - \lambda A)^{-1}$ is a function of exponential type. The type of growth of the resolvent $(I - \lambda A)^{-1}$ we shall agree to call the type of the operator A and denote by $\sigma(A)$. If $\sigma(A) = 0$, then $A = 0$ ([1]).

It will be shown below that for every operator $A \in \Lambda^{(\exp)}$ there exists an orthogonal resolution of the identity $P(x)$ $(0 \leq x \leq \sigma(A))$ such that, in the sense of uniform convergence,

$$A = 2i \int_0^{\sigma(A)} P(x) A_I \, dP(x). \tag{1}$$

Representations of the form (1) are well known ([2,3]) for the class of all completely continuous operators satisfying condition 1). Operators of the class $\Lambda^{(\exp)}$, as is easy to see, are not all completely continuous.

1. We shall say that a function of the complex variable $W(\lambda)$, whose values are bounded linear operators acting in a separable Hilbert space $\mathfrak{G}$, belongs to the class $\Omega_{\mathfrak{G}}^{(\exp)}$ if: I) $W(\lambda)$ is an entire function of exponential type; II) $W^*(\lambda)W(\lambda) - I = 0$ $(\operatorname{Im}\lambda = 0)$; III) $W^*(\lambda)W(\lambda) - I \geq 0$ $(\operatorname{Im}\lambda < 0)$; IV) $W(0) = I$. The type of the function $W(\lambda)$ will be denoted by $\sigma(W)$.

For a given operator $A \in \Lambda^{(\exp)}$ one can, obviously, construct a bounded linear operator K, acting from some separable Hilbert space $\mathfrak{G}$ into $\mathfrak{H}$, so that the equality $KK^* = A_I$ holds. The collection

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

is called an **exponential node**, and the operator-function

$$W_\theta(\lambda) = I + 2i\lambda K^*(I - \lambda A)^{-1}K \tag{2}$$

the **characteristic function** of the node θ. Direct verification shows that $W_\theta(\lambda) \in \Omega_{\mathfrak{G}}^{(\exp)}$. Conversely, every function $W(\lambda) \in \Omega_{\mathfrak{G}}^{(\exp)}$ is characteristic for some exponential node ([4]). If $W(\lambda)$ is the characteristic function of the exponential node

$$\begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix},$$

then $\sigma(W) = \sigma(A)$ ([4]).

Let

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

be an exponential node and $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Define in $\mathfrak{H}_j$ $(j = 1, 2)$ the operator A_j, putting $A_{jh} = P_{jAh}$ $(h \in \mathfrak{H}_j)$, where P_j is the orthoprojector onto $\mathfrak{H}_j$. If the subspace $\mathfrak{H}_1$ is invariant with respect to A, then

$$\theta_j = \begin{pmatrix} A_j & P_{jK} \\ \mathfrak{H}_j & \mathfrak{G} \end{pmatrix} \quad (j = 1, 2)$$

are exponential nodes and the formula $W_\theta(\lambda) = W_{\theta_1}(\lambda)W_{\theta_2}(\lambda)$ holds ([4]). The node θ_j is called the projection of the node θ onto the subspace $\mathfrak{H}_j$.

2. We shall need the following properties of operators of the class $\Lambda^{(\exp)}$.

a) *If $A \in \Lambda^{(\exp)}$ and $0 < \sigma_0 < \sigma(A)$, then there exists a subspace $\mathfrak{H}_0$ invariant with respect to A such that the operator A_0 induced in it satisfies the condition $\sigma(A_0) = \sigma_0$.*

Indeed, there exists a subspace $\mathfrak{H}_1$ invariant with respect to A, in which a unicellular operator A_1 is induced, whose type coincides with the type of the operator A ([1]). From the unicellularity of the operator A_1 it follows, as was shown in ([4]), that it is completely continuous and has a nuclear imaginary component. By virtue of the criterion of unicellularity ([5]),

$$\sigma(A) = 2\,\mathrm{sp}(A_1)_I.$$

It remains to note that the traces of the imaginary components of operators induced in invariant subspaces of the operator A_1 take all values from 0 to $\mathrm{sp}(A_1)_I$ ([6]).

b) *If* $A \in \Lambda^{(\exp)}$, *then* $\|A\| \leqslant \sigma(A)$.

Let us suppose first that the operator A is unicellular. Then it is completely continuous ([4]) and is representable in the form (1). Let

$$0 = x_0 < x_1 < \dots < x_n = \sigma(A)$$

be some partition of the segment $[0, \sigma(A)]$. Consider an orthonormal basis $\{e_\alpha\}_1^\infty$, consisting of eigenvectors of the operator A_I, and denote by ω_α the eigenvalues corresponding to the vectors e_α. Since

$$\left|\left(\sum_{j=1}^{n} P(x_j) A_I \Delta P_j f, g\right)\right| \leqslant \sum_{j=1}^{n} \sum_{\alpha=1}^{\infty} \omega_\alpha \left|(\Delta P_j f, e_\alpha)\right| \left|(P(x_j) e_\alpha, g)\right| \leqslant$$

$$\leqslant \|g\| \sum_{\alpha=1}^{\infty} \omega_\alpha \sum_{j=1}^{n} \left|(\Delta P_j f, e_\alpha)\right| \leqslant \|f\| \, \|g\| \operatorname{sp} A_I \qquad (\Delta P_j = P(x_j) - P(x_{j-1})),$$

then

$$\|A\| \leqslant 2 \operatorname{sp} A_I \ (= \sigma(A)).$$

In the case when A is not unicellular, for an arbitrary vector $f \in \mathfrak{H}$ we construct the subspace $\mathfrak{H}_f$, which is the closure of the linear span of the vectors $A^n f$ $(n = 0, 1, \dots)$. The operator A_f, induced in the subspace $\mathfrak{H}_f$, is unicellular ([7]). Consequently, $\|A_f\| \leqslant \sigma(A_f)$, and since $\sigma(A_f) \leqslant \sigma(A)$, then

$$\|Af\| = \|A_f f\| \leqslant \sigma(A) \|f\| \quad (f \in \mathfrak{H}),$$

and hence $\|A\| \leqslant \sigma(A)$.

3. Let $A \in \Lambda^{(\exp)}$ and let $\mathfrak{H}_\gamma$ $(\gamma \in \Gamma)$ be all possible subspaces invariant with respect to A, in which operators are induced whose types do not exceed some fixed value $x \in (0, \sigma(A)]$.

Embed the operator A in the node

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

and denote by θ_γ and θ_x the projections of the node θ onto $\mathfrak{H}_\gamma$ and onto the closure $\mathfrak{H}_x$ of the linear span of the subspaces $\mathfrak{H}_\gamma$ $(\gamma \in \Gamma)$. According to one of the theorems of ([8]),

$$\sigma(W_{\theta_x}) = \sup_{\gamma \in \Gamma} \sigma(W_{\theta_\gamma}).$$

Denoting by A_x the operator induced in $\mathfrak{H}_x$, we obtain

$$\sigma(A_x) = \sigma(W_{\theta_x}) = \sup_{\gamma \in \Gamma} \sigma(A_\gamma) = x.$$

Lemma. *If P_Δ is the orthoprojector onto the subspace*

$$\mathfrak{H}_\Delta = \mathfrak{H}_{x_2} \ominus \mathfrak{H}_{x_1},$$

$(0 < x_1 < x_2 \leqslant \sigma(A))$ *and* $A_\Delta f = P_\Delta A f$ $(f \in \mathfrak{H}_\Delta)$, *then* $\sigma(A_\Delta) = x_2 - x_1$.

Proof. Denoting by θ_Δ the projection of the node θ_{x_2} onto $\mathfrak{H}_\Delta$, we obtain

$$W_{\theta_2}(\lambda) = W_{\theta_1}(\lambda) W_{\theta_\Delta}(\lambda).$$

Consequently,

$$\sigma(W_{\theta_2}) \leqslant \sigma(W_{\theta_1}) + \sigma(W_{\theta_\Delta}),$$

so that

$$\sigma(A_\Delta) \geqslant x_2 - x_1.$$

Fix a vector $f \in \mathfrak{H}_\Delta$ and consider the operator A_f induced in $\mathfrak{H}_f$. Obviously,

$$x_1 < \sigma(A_f) \leqslant x_2.$$

Let P_f be the orthoprojector onto the subspace

$$\mathfrak{G}_f = \mathfrak{H}_f \ominus (\mathfrak{H}_f \cap \mathfrak{H}_{x_1}),$$

and

$$B_f h = P_f A_f h \qquad (h \in \mathfrak{G}_f).$$

Since the operator induced in $\mathfrak{H}_f \cap \mathfrak{H}_{x_1}$ has type x_1, it follows that

$$\mathrm{sp}(B_f)_I \leqslant \tfrac{1}{2}(x_2 - x_1).$$

Introduce the operators T and C_f, of which the first assigns to each vector of $\mathfrak{H}_f$ its orthogonal projection onto $\mathfrak{H}_\Delta$, while the second is induced by the operator A_Δ in the subspace

$$\mathfrak{K}_f = T\mathfrak{H}_f$$

invariant for it. The operator

$$A_\Delta T (= T A_f)$$

is completely continuous. Consequently, there exist orthonormi-

orthonormal sequences $\{\varphi_\alpha\}_1^\nu$ $(\nu \leq \infty)$ and $\{\psi_\alpha\}_1^\nu$, belonging respectively to the subspaces $\mathfrak{H}_f$ and $\mathfrak{H}_\Delta$, such that $A_\Delta T h = \sum_{\alpha=1}^{\nu} (h, \varphi_\alpha)\omega_\alpha \psi_\alpha$ $(h \in \mathfrak{H}_f,\ \omega_\alpha > 0)$. It is not hard to see that $\{\varphi_\alpha\}_1^\nu$ and $\{\psi_\alpha\}_1^\nu$ are bases of the subspaces $\mathfrak{H}_f$ and $\mathfrak{R}_f$. From the equalities

$$(A_f\varphi_\alpha, \varphi_\alpha) = \frac{1}{\omega_\alpha}\left(\sum_{\beta=1}^{\nu}(A_f\varphi_\alpha, \varphi_\beta)\omega_\beta, \psi_\alpha\right) = \frac{1}{\omega_\alpha}(A_\Delta T A_f \varphi_\alpha, \psi_\alpha) = (A_\Delta \psi_\alpha, \psi_\alpha)$$

it follows that

$$\operatorname{sp}(C_f)_I = \sum_{\alpha=1}^{\nu}((C_f)_I\psi_\alpha, \psi_\alpha) = \sum_{\alpha=1}^{\nu}((A_\Delta)_I\psi_\alpha, \psi_\alpha) = \sum_{\alpha=1}^{\nu}((A_f)_I\varphi_\alpha, \varphi_\alpha) =$$

$$= \sum_{\alpha=1}^{\nu}((B_f)_I\varphi_\alpha, \varphi_\alpha) \le \frac{1}{2}(x_2 - x_1).$$

For every operator $A \in \Lambda^{(\exp)}$ the inequality $\sigma(A) \le 2 \operatorname{sp} A_I$ holds. Consequently, $\sigma(C_f) \le x_2 - x_1$. Since $\mathfrak{H}_\Delta$ is the closure of the linear span of all subspaces $\mathfrak{R}_f$ $(f \in \mathfrak{H}_\Delta)$, it follows that $\sigma(A_\Delta) \le x_2 - x_1$.

The lemma is proved. Denote by $P(x)$ $(0 < x \le \sigma(A))$ the orthogonal projector onto $\mathfrak{H}_x$, and put $P(0) = 0$. We shall call the function $P(x)$ $(0 \le x \le \sigma(A))$ the extremal spectral function of the operator A. It is easy to show that $P(x)$ is continuous on the interval $(0, \sigma(A)]$. If A is a completely non-self-adjoint operator ([4]), then it is also continuous at the point 0.

Theorem 1. *An operator $A \in \Lambda^{(\exp)}$ admits a norm-convergent triangular representation (1), where $P(x)$ is its extremal spectral function.*

Proof. It is enough to show ([2]) that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, if the partition $0 = x_0 < x_1 < \cdots < x_n = \sigma(A)$ satisfies the condition $x_j - x_{j-1} < \delta$, then

$$\left\| \sum_{j=1}^{n} \Delta P_j A \Delta P_j \right\| < \varepsilon$$

$(\Delta P_j = P(x_j) - P(x_{j-1}))$. Put $\delta = \varepsilon$. Using assertion b) and the lemma, we obtain:

$$\left\| \sum_{j=1}^{n} \Delta P_j A \Delta P_j \right\| = \max_j \|\Delta P_j A \Delta P_j\| \le \max_j \sigma(\Delta P_j A \Delta P_j) < \varepsilon.$$

Theorem 2. *If $W(\lambda) \in \Omega_{\mathfrak{G}}^{(\exp)}$, then in the sense of uniform convergence*

$$W(\lambda) = \int_0^{\sigma(W)} e^{i\lambda x}\, dF(x),$$

where $F(x)$ is a strictly increasing operator-function satisfying the condition

$$\|F(x') - F(x'')\| \le |x' - x''|.$$

Proof follows from formula (1) with the aid of arguments analogous to those given in work ([9]).

Theorem 2 is a special case of a more general assertion obtained by Yu. P. Ginzburg ([10]) by a purely analytic method.

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Note: Figure translations are in progress. See original paper for figures.