THE NONSTATIONARY HEAT-CONDUCTION PROBLEM FOR A TWO-LAYER CYLINDER

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THE NONSTATIONARY HEAT-CONDUCTION PROBLEM FOR A TWO-LAYER CYLINDER

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1. The nonstationary axisymmetric problem of heat propagation in two infinite coaxial cylinders with differing physical properties is solved.* The thermal contact between the cylinders is assumed to be ideal. The temperature in the system satisfies the equation

$$Lu = \partial u/\partial \tau, \tag{1}$$

where the linear operator L is defined as $L = a_\beta \nabla^2$; $\beta = 1, 2$.

$$u(\tau, R_2) = 0; \tag{2}$$

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial r} \left[u(\tau, R_1 - \varepsilon) - \xi u(\tau, R_1 + \varepsilon) \right] = 0; \tag{3}$$

a_β is the thermal diffusivity coefficient; R_β are the radii of the cylinders; $\xi = \lambda_1/\lambda_2$ is the ratio of the thermal-conductivity coefficients; index 1 refers to the inner region $0 \le r \le R_1$, and index 2 to the outer region $R_1 \le r \le R_2$. After separation of variables the solution is written in the form

$$u(\tau, r) = \sum_i c_i \psi_i(r) \exp(-s_i \tau), \tag{4}$$

where $L\psi_i = s_i \psi_i$, and c_i are the coefficients in the expansion of the initial temperature field $u(0, r)$ in $\{\psi_i\}$. Obviously, the relaxation properties of the system are determined by the minimum eigenvalue s_i. The eigenfunctions in regions 1 and 2 are linear combinations of Bessel functions of the first and second kind,

$$\psi_i(\beta) = A_i^{(\beta)} J_0\left(r\sqrt{s_i/a_\beta}\right) + B_i^{(\beta)} N_0\left(r\sqrt{s_i/a_\beta}\right),$$

and from the boundedness of the solution it follows that $B_i^{(1)} = 0$, while from the boundary conditions (2), (3) and the continuity of the eigenfunctions it follows that the eigenvalues s_i are the roots of the transcendental equation

$$[z\xi J_1(xyz)N_0(xy) - J_0(xyz)N_1(xy)]J_0(x) - [z\xi J_1(xyz)J_0(xy) - J_0(xyz)J_1(xy)]N_0(x) = 0, \tag{5}$$

where, for convenience, the dimensionless variables have been introduced

$$x = \sqrt{s_i/a_2}\, R_2, \qquad y = R_1/R_2, \qquad z = \sqrt{a_2/a_1}.$$

Equation (5) has an infinite number of roots, whose determination is associated with laborious computations. In the cases $y = 0$ and $y = 1$ equation (5) takes the form $J_0(x)$ and $J_0(xz) = 0$,

$$s_k = \alpha_k^2 a_2/R_2^2, \qquad s_k = \alpha_k^2 a_1/R_2^2, \tag{6}$$

* This problem is in many respects analogous to the problem of the propagation of electromagnetic waves in a waveguide partially filled with a dielectric.

where a_k are the zeros of the function $J_0(a)$. It is easy to see that this is the solution of the well-known heat-conduction problem in a cylinder of radius R_2. In the present paper approximate solutions and a qualitative analysis of the dependence of the eigenvalues x_k on the physical parameters of the problem are given.

2. Consider (5) in the region $y \ll 1$, assuming $x \gg 1$,* but $xy \ll 1$, $xyz \ll 1$ (the boundaries of this region are indicated below). Using the asymptotic expressions for the Bessel functions, in this region we obtain

$$\operatorname{tg}(x - \pi/4) \simeq 2\ln(\gamma xy/2) + 8/\pi\,(\xi z^2 - 1)x^2y^2 \tag{7}$$

or

$$x_k = \pi(k - {}^3\!/_4) + \operatorname{arc\,tg}\,[2\ln(\gamma xy/2) + 8/\pi\,(\xi z^2 - 1)x^2y^2], \tag{8}$$

where $\gamma = \exp(0.57772) = 1.7811$. As $y \to 0$,

$$x_k = \pi({}^3\!/_4 + k)\left[1 - \frac{\pi^2(3 + 4k)}{2^5}(\xi z^2 - 1)y^2\right], \tag{8a}$$

i.e., there exists a finite region in y, the size of which is determined by ξz^2, in which the presence of the inner cylinder practically does not affect the characteristic relaxation time of the system. (We note that from (5) it follows that

$(dx/du)_0 = 0$.) We shall further assume that the ratio of the volumetric heat capacities $\xi z^2 > 1$. Since in (8) the arctangent is a function with bounded variation, we investigate its argument. It has a minimum at

$$xy = 2^{3/2}\pi^{1/2}(\xi z^2 - 1)^{-1/2}, \tag{8b}$$

so that

$$\pi(k - {}^3/_4) + \operatorname{arc\,tg}\left[1 + \ln(2\gamma^2/(\xi z^2 - 1))\right] \le x_k \le \pi(k + {}^1/_4),$$

and in the region under consideration the inequality $x \gg 1$ can be violated only for x_1. We shall therefore confine ourselves for the time being to the case

$$1 \le \xi z^2 \le 1 + 2e\gamma^2/\pi = 6.4896, \tag{9}$$

when $x_1 \ge \pi/4$ and the violation of the inequality $x \gg 1$ is not very substantial.

Thus, the minimum eigenvalue x_1, which, as indicated, determines the relaxation properties of the system, decreases with increasing y in accordance with (8), while the product xy increases and may become greater than 1. In this connection the region of parameter values z and ξ specified by inequality (9) splits into two:

$$\xi z^2 \le 1 + 8/\pi = 3.5464; \tag{10a}$$

$$1 + 8/\pi \le \xi z^2 \le 1 + 2e\gamma^2/\pi. \tag{10b}$$

3. In the first of these, the inequality $xy < 1$ is violated before $x_1(y)$ reaches a minimum. This occurs at the point

$$y_1 = \{\pi/4 + \operatorname{arc\,tg}[2\ln(\gamma/2) + 8/\pi\,(\xi z^2 - 1)]\}^{-1}, \tag{11}$$

where $4/3\pi \le y_1 \le 0.69$.

For $y > y_1$ the product $xy \ge 1$, and equation (5) is approximately written in the form

$$\xi z^2 xy \sin x(1-y) \simeq 4\cos(1-y)x \tag{12}$$

or

$$x_k = [\pi(k-1) + \operatorname{arc\,tg}(4/\xi z^2 xy)]/(1-y). \tag{12a}$$

Taking (10a) into account, the arctangent here may be replaced by its argument, so that

$$x_1(y) \simeq 2[\xi z^2 y(1-y)]^{-1/2}. \quad (13)$$

Expression (13) is valid for $y \geq y_2$, where y_2 is determined from (12a) and the condition $xy_2 = 1$:

$$0.543 = (1 + \operatorname{arc\,tg}(\pi/(\pi+8)))^{-1} \leq y_2 \simeq 1/[1 + \operatorname{arc\,tg}(4/\xi z^2)] \leq$$

$$\leq (1 + \operatorname{arc\,tg}(\pi/(\pi + 2e\gamma^2)))^{-1} = 0.645.$$

* We note that, with good accuracy, $a_k \gg 1$, since

$$a_k = \pi(k - {}^1/_4) + \frac{1}{2\pi(4k+1)} - \frac{31}{6\pi^3(4k+1)^3} + \frac{3779}{15\pi^5(4k+1)^5} + \cdots$$

It is easy to verify that $y_1 \simeq y_2$, and for values of y close to them (8) and (13) almost coincide, but, of course, (8) is a lower estimate and (13) an upper estimate describing the region near the minimum of the function $x_1(y)$, located at $y_m = 1/2$ and equal to $x_m = 4(\xi z)^{-1/2}$. (13) remains valid up to the point

$$y_3 = [1 + z \operatorname{arc\,tg}(4/z\xi)]^{-1} \simeq \xi/(4+\xi). \quad (14)$$

4. To the right of this point $xyz \to 1$, and therefore the solution near $y = 1$ can be found in the form of a series in powers of $(1-y)$. Differentiating (5), we find

$$x_k(y, z, \xi) = x_k(1, z)\{1 - (\xi - 1)(1-y) + ...\}. \quad (15)$$

5. Let us now consider the parameter region (10b). For $0 \leq y \leq y_2$ the function $x_1(y)$, according to (8), decreases, passes through a minimum, which is deeper than in item 3, and begins to increase up to the point y_1, to the right of which $x_1(y)$ is approximated by the increasing dependence (13), and further—as in item 3.

6. Let us now turn to the parameter region $\xi z^2 > 1 + 2e\gamma/\pi$ (for simplicity we assume that the inequality is satisfied with a large margin), when the function $x_1(y)$ may become less than unity. Taking into account the approximate nature of the analysis, put in (8) $x_1 = \pi/4$, and not 1. Then, solving an equation of the form $u \ln u = -v$, which arises when the argument of the arctangent in (8) is set equal to zero, we find the point

$$y_1 \simeq 2^{7/2}\pi^{-3/2}[(\xi z^2-1)\ln(\pi(\xi z^2-1)/2\gamma^2)]^{-1/2}, \tag{16}$$

to the left of which the solution behaves according to (8)*. For $y > y_1$, assuming $x \leq 1$, instead of (5) we have

$$\xi z^2 xy\ln(\gamma xy/2) - (\xi z^2-1)xy\ln(\gamma x/2) + 4/xy = 0. \tag{17}$$

(17) is also reduced to the form $u\ln u = -v$, and the solution has the form

$$x_1 = 2^{3/2}y^{-1}\{2(\xi z^2-1)\ln(1/y) - \ln 2\gamma^2\}^{-1/2}. \tag{18}$$

The function $x_1(y)$ reaches its minimum

$$x_m = 2(\xi z^2-1)^{-1/2}\exp[1+\ln 2\gamma^2/2(\xi z^2-1)] \tag{19}$$

at the point

$$y_m = \exp[-1-\ln(2\gamma^2)/2(\xi z^2-1)]. \tag{19a}$$

The approximate solution (18) is valid inside the interval $y_2 \leq y \leq y_3$, where $x \leq 1$. Putting $x = 1$ in (17), we find

$$y_2 \simeq 2^{3/2}[\xi z^2\ln(\xi z^2/8) - \ln(2/\gamma)^2]^{-1/2}, \tag{20}$$

which almost coincides with (16), and

$$y_3 \simeq (\gamma/2)^{1/\xi z^2}\{1-(2/\gamma)^{2/\xi z^2}\cdot 8/\xi z^2\}^{1/2} \simeq 1 - 3\ln(2/\gamma)/\xi z^2. \tag{21}$$

To the right of the point y_3 the condition $x < 1$ is violated (and almost immediately the condition $xy < 1$ as well), and therefore on the comparatively narrow interval $[y_3, 1]$, for $x_1(y)$ one should use the approximation (15).

7. The analysis carried out above refers to the case $z < 1$; however, before passing to the case $z > 1$, let us discuss the already revealed dependence of the solution on ξ and z. It is easy to see that for $y < y_3$ (where y_3 is determined from (14) and (21)), $x_1(y, \xi, z)$ depends only on the combination ξz^2, i.e., in a sufficiently broad region the relaxation properties of the system do not depend on the thermal conductivity of the inner cylinder, and its influence is manifested only

Fig. 1. $x = f(y)$, $z \gg 1$, $\xi > 4$ ($z = 3$, $\xi z^2 = 50$). The numbers of the formulas for $x_1(y)$ in each region of y are indicated in parentheses.

Figure 1: Fig. 1. $x = f(y)$, $z \gg 1$, $\xi > 4$ ($z = 3$, $\xi z^2 = 50$). The numbers of the formulas for $x_1(y)$ in each region of y are indicated in parentheses.

* The indicated equation also has a second root

$$y_1' \simeq \frac{8}{\pi\gamma}\left[1 - \frac{2\gamma^2}{\pi(\xi z^2 - 1)}\right]^{1/2},$$

which is greater than 1 by virtue of $\xi z^2 - 1 > 2e\gamma^2/\pi$.

through the ratio of the volume heat capacities of the cylinders. Moreover, taking into account the assumption made earlier, $xyz \ll 1$, it can be shown that (10a, b) must be supplemented by the inequalities $\xi z^2 \gg 4z^2$ and $\xi z^2 \gg 1{+}2.2z^2$, as a result of which the interval $[y_3, 1]$, where s_1 depends not on the ratio of the volume heat capacities ξz^2, but on the ratio of the thermal conductivities ξ and the temperature diffusivity inside the cylinder, is very narrow.

Fig. 1. $x = f(y)$, $z \gg 1$, $\xi > 4$ ($z = 3$, $\xi z^2 = 50$). The numbers of the formulas for $x_1(y)$ in each region of y are indicated in parentheses.

8. Thus, let us consider (15) for $z > 1$. It is evident in advance that for $z \sim 1$, $x_1(y)$ is a smooth curve with a minimum, whose depth depends on ξ, while $x_1(0)$ and $x_1(1)$ are approximately equal. It is clear that for small ξz^2, over a wide range of y, the difference between $x_1(y)$ and $x_1(0)$ is small, and therefore it is expedient to seek a solution only for $z \gg 1$ and $\xi z^2 \gg 1$.

As before, for small y the solution decreases according to (8) up to the point y_1, where the condition $x > 1$ is violated. However, it is now necessary to check where the assumption $xyz < 1$ is violated. Substituting $xy = 1$ in (7), we find, with allowance for $\xi z^2 \gg 1$,

$$\tilde{y}_1 = z^{-1}\left[\pi/4 + \operatorname{arc\,tg}(2\ln(\gamma/2z)) + 8/\pi\xi\right]^{-1}. \tag{22}$$

We shall not solve the inequality $y_1 \leq \tilde{y}_1$, since from (22) it is easy to obtain that under the condition

$$z > \frac{1}{2\gamma}\exp(-1/2 - 4/\pi\xi) \simeq 1$$

$\tilde{y}_1$ becomes negative, so that the transition from solution (8) to (18) for $z > 1$ is analogous to that already considered in Sec. 6. Accordingly, the conclusions drawn earlier concerning the dependence of x_1 on the parameters of the system remain valid. In the region $y > y_1$, the function $x_1(y)$ decreases, and, depending on the magnitude of ξ, two cases are possible. To examine them, from (17) we find the point where $xyz = 1$,

$$\tilde{y}_3 \simeq \exp\left[-(8z^2 + \ln 2\gamma^2)/2\xi z^2\right]. \tag{23}$$

Comparing $\tilde{y}_3$ with (19), it is easy to see that for $\xi z^2 \geq 4$, $y_1 < \tilde{y}_3$, i.e. the minimum of the function $x_1(y)$ is described according to (18), (19). For $y > y_3$, instead of (5), in view of $xyz > 1$, we have

$$\operatorname{tg}\left(xyz - \frac{\pi}{4}\right) = \frac{1}{\xi z \ln y}\left(-\frac{1}{xy} - xy \ln\frac{\gamma x}{2}\right) \tag{24}$$

or

$$x_1 = \left[\frac{\pi}{4} + \operatorname{arctg}\left\{\frac{1}{\xi z \ln y}\left(-\frac{1}{xy} - xy\ln\frac{\gamma x}{2}\right)\right\}\right] \Big/ yz; \tag{25}$$

since $x \ll 1$ and $1 - y_3 \ll 1$, (26) can be rewritten in the form

$$x_1(y) \simeq \left[\frac{\pi}{4} + \operatorname{arctg}\left(\frac{4}{3\pi\xi\ln(1/y)}\right)\right] \Big/ yz; \tag{25a}$$

(25) as $y \to 1$ agrees well with (6), and, moreover, from (24) we have

$$(dx/dy)_{y=1} = x_1(1)\left[\xi - 1 - x\ln(\gamma x/2)\right], \tag{26}$$

which differs from the exact result (15) by a term of order $2/\gamma e(\xi - 1) \ll 1$. Finally, putting $xyz = 1$ in (24), we find the left boundary of the range of applicability of (26), (25),

$$\tilde{y}_4 \simeq \exp\left[-(z^2 - \ln z)/\xi z^2(1 - \pi/4)\right], \tag{27}$$

which agrees well with $\tilde{y}_3$. The general behavior of the function $x_1(y)$ for $z \gg 1$, $\xi > 4$ is shown in Fig. 1.

If $1 \leq \xi \leq 4$, then the dependence $x_1(y)$ is analogous to the preceding case, with the only difference that the region of the minimum of the function is described not by expressions (17), (18), but by (24), (25).

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