MAGNITUDES OF DEVIATIONS OF MEROMORPHIC FUNCTIONS OF LOWER ORDER LESS THAN ONE

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 517.535.6

MATHEMATICS

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MAGNITUDES OF DEVIATIONS OF MEROMORPHIC FUNCTIONS OF LOWER ORDER LESS THAN ONE

(Presented by Academician M. A. Lavrent' ev on 29 XI 1968)

§ 1. Consider a function $f(z)$, meromorphic for $z \neq \infty$, of finite lower order λ and order ρ. In [1] the properties were considered of the quantities

$$\beta(a; f) = \lim_{r\to\infty} \frac{\ln^+ M(r,a,f)}{T(r,f)},$$

where a is an arbitrary complex number from the extended complex plane,

$$M(r,a,f) = \max_{|z|=r} \frac{1}{|f(z)-a|}, \qquad a \neq \infty,$$

$$M(r,\infty,f) = \max_{|z|=r} |f(z)|,$$

and $T(r,f)$ is the Nevanlinna characteristic of $f(z)$.

It turned out (see [1]) that the quantities $\beta(a,f)$ for meromorphic functions of finite lower order possess a number of properties analogous to the properties of the deficiency quantities of R. Nevanlinna [2].

In particular, the following assertion is valid:

Theorem A [1]. *For an arbitrary meromorphic function $f(z)$ of finite lower order λ, the set*

$$\Omega(f) = \{a: \ \beta(a,f) > 0\}$$

is at most countable and

$$\sum_{(a)} \beta(a,f) \le K_1(\lambda+1),$$

where K_1 is an absolute constant.

Definition. We shall call the quantity $\beta(a, f)$ the **magnitude of deviation** of the meromorphic function $f(z)$ from the number a. If for some number a, $\beta(a, f) > 0$, then we shall say that $f(z)$ has a **positive deviation** from the number a.

It follows from Theorem A that the magnitudes of deviations of meromorphic functions of finite lower order are equal to zero for almost all numbers a from the extended complex plane.

§ 2. The deficiency quantities and other asymptotic properties of meromorphic functions of lower order $\lambda < 1$ have been studied by many authors [3-6].

We note the following important assertion [3, 4].

Theorem B. *If $f(z)$ is a meromorphic function of lower order $0 \le \lambda < 0.5$ and, for some a,*

$$\delta(a, f) > 1 - \cos \pi\lambda,$$

then

$$\lim_{r\to\infty} \mu(r, a, f) = \infty,$$

$$\mu(r, a, f) = \min_{|z|=r} \frac{1}{|f(z) - a|}.$$

It follows from this theorem that if the lower order of $f(z)$ is $\lambda < 0.5$ and $f(z)$ has no fewer than two deficient values, then all the magnitudes of the deficiencies are sufficiently small. Stronger results in this direction are known (see ([3],[5])).

In the present work we obtain the following assertions on the quantities $\beta(a, f)$ for meromorphic functions of finite lower order $\lambda < 1$, which are certain analogues of the preceding results.

Theorem 1. *If, for a meromorphic function $f(z)$ of finite lower order $0 \le \lambda < 0.5$, for some a*

$$\beta(a, f) > \pi\lambda \sin \pi\lambda$$

or

$$\beta(a, f) > \pi\lambda \tan \frac{\pi\lambda}{2} \cdot (2 - \delta(a, f)),$$

then

$$\overline{\lim}_{r\to\infty}\mu(r,a,f)=\infty,$$

and, consequently,

$$\beta(b,f)=0 \text{ and } \delta(b,f)=0$$

for all $b\neq a$.

Remark. *The deviations of $f(z)$ from all numbers $b\neq a$ are also equal to zero if the weaker relations* $(0\leq\lambda\leq 0.5)$

$$\beta(a,f)\geq\pi\lambda\sin\pi\lambda$$

or

$$\beta(a,f)\geq\pi\lambda\tan\frac{\pi\lambda}{2}\cdot[2-\delta(a,f)]$$

are satisfied.

Theorem 2. *If a meromorphic function $f(z)$ of finite lower order $\lambda<1$ has positive deviations from at least two points a and b, then*

$$\beta(a,f)+\beta(b,f)\leq\pi\lambda\tan\frac{\pi\lambda}{2}\cdot[2-\delta(a,f)-\delta(b,f)],$$

$$\beta(a,f)+\cos\pi\lambda\,\beta(b,f)\leq\pi\lambda\sin\pi\lambda\cdot[1-\delta(b,f)]\quad(0\leq\lambda<0.5).$$

The example analyzed in the work of A. Edrei and W. Fuchs ([5]) shows that the assertions of Theorems 1 and 2 cannot be strengthened.

§ 3. The proofs of Theorems 1 and 2 are carried out by means of the method used by us in ([1]). Indeed, let, for fixed r, $\theta(r)$ and $\theta^*(r)$ be defined by the condition $(0\leq\theta(r)<2\pi,\ 0\leq\theta^*(r)<2\pi)$

$$|f(re^{i\theta(r)})|=M(r,\infty,f),\qquad\frac{1}{|f(re^{i\theta^*(r)})|}=M(r,0,f).$$

For fixed r, together with the meromorphic function $f(z)$ $(f(0)=1)$, consider the meromorphic functions

$$F_r(z)=f(r^{i\theta(r)}z),\qquad F_r^*(z)=\frac{1}{f(e^{i\theta^*(r)}z)}\quad(z=se^{i\theta}).$$

For each fixed r ($r_0 \leq r \leq 0.5R$) we have (see ([1]))

$$\ln M(r,\infty,f)+\ln M(r,0,f)\leq (2x)^2 r^{2x}\int_0^R\left\{\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\ln|F_r(te^{i\theta})|\,d\theta+ +\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\ln|F_r^*(te^{i\theta})|\,d\theta\right\}\frac{t^{2x-1}\,dt}{(t^{2x}+r^{2x})^2}+\sum_{|b_k|\leq 2R}\ln\left|\frac{r^{2x}+|b_k|^{2x}}{r^{2x}-|b_k|^{2x}}\right|+ +\sum_{|a_k|\leq 2R}\ln\left|\frac{r^{2x}+|a_k|^{2x}}{r^{2x}-|a_k|^{2x}}\right|+C_1\left(\frac{r}{R}\right)^{2x}\{T(4R,f)+T_1(4R,f)\}; \tag{1}$$

where $x=\pi/2\alpha$, $\pi-\varepsilon<\alpha=\alpha(r)<\pi$ $(0<\varepsilon<\pi/2)$, a_k are the zeros of the meromorphic function $f(z)$, and b_k are its poles (C_1 is a positive constant).

Moreover $(0\leq t\leq R)$,

$$\int_{-\pi}^{\pi}\ln|F_r(te^{i\theta})|\,d\theta+\int_{-\pi}^{\pi}\ln|F_r^*(te^{i\theta})|\,d\theta=0. \tag{2}$$

Theorem 2 follows from estimate (1), if one uses relation (2) and carries out arguments analogous to those given in papers ([1],[7],[8]). **Theorem 1** is proved analogously.

Theorems 1 and 2 can also be obtained by means of the method used in papers ([3],[9]).

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Received
12 XI 1968

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Note: Figure translations are in progress. See original paper for figures.