ON THE THEORY OF LINEAR PASSIVE SYSTEMS

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Abstract

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1. Introduction. By a **linear passive system** we shall mean a system of convolution equations

$$Z * u = f, \tag{1}$$

where $f = (f_1, \dots, f_N)$ is a given vector distribution from $(\mathcal{D}')^{\times N} = \mathcal{D}' \times \cdots \times \mathcal{D}'$, $u = (u_1, \dots, u_N)$ is an unknown element of $(\mathcal{D}')^{\times N}$, and $Z = \|Z_{kj}\|$ is an $N \times N$ matrix with material elements $Z_{kj}(\xi)$ from $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$, satisfying the conditions:

a) **causality** with respect to some convex closed acute cone $\Gamma \subset \mathbb{R}^n$ with vertex at O: $Z(\xi) = 0,\ \xi \notin \Gamma$;

b) **passivity**: for all complex $a \in \mathbb{C}^N$, $\varphi \in \mathcal{D}$, and $q \in C = \operatorname{int} \Gamma^*$, the inequality

$$\operatorname{Re} \int [(Z e^{-(q,\xi)} a, a) * \varphi](x)\, \overline{\varphi}(x) \geq 0; \tag{2}$$

holds; here Γ^* is the cone conjugate to the cone Γ.

The operator $Z*$, defining the linear passive system (1), will be called a **passive operator** (with respect to the cone Γ), and the corresponding matrix $Z(\xi)$ the (generalized) **impedance** of the system.

A number of problems of mathematical physics reduce to linear passive systems: linear thermodynamic systems, the theory of electric circuits, scattering of nuclear particles, scattering of electromagnetic waves, etc. ([1-8]). Inequality (2) reflects the ability of a physical system to absorb energy, but not to generate it.

Linear passive systems have been studied in especially great detail in the case of one independent variable ([1-4]). The principal apparatus of investigation in this case is the classical integral representation of functions holomorphic in the upper half-plane and having there nonnegative real part ([9]). Using the results of

([10]) on the representation of holomorphic functions of many complex variables with nonnegative real part in a tubular domain over a cone, we have investigated linear passive systems in the multidimensional case. Here we present the most important results of these investigations.

2. Characteristic of a passive operator. From conditions a) and b) it follows that $Z_{kj} \in \mathcal{S}'(\Gamma)$, so that the matrix $Z(\xi)$ has a Fourier–Laplace transform ([11]).

Theorem 1. In order that an $N \times N$ matrix $Z(\xi)$ define a passive operator with respect to the cone Γ, it is necessary and sufficient that its Fourier–Laplace transform $\widetilde{Z}(\zeta)$, $\zeta = p + iq$, be a holomorphic matrix-function in the tubular domain $T^C = \mathbb{R}^n + iC$, $C = \operatorname{int}\Gamma^*$, satisfying the relation

$$\widetilde{Z}(p + iq) = \overline{\widetilde{Z}(-p + iq)}, \qquad \zeta \in T^C, \tag{3}$$

and the inequality

$$\operatorname{Re}(\widetilde{Z}(\zeta)a, a) \geq 0, \qquad a \in \mathbb{C}^N, \quad \zeta \in T^C. \tag{4}$$

Remark. The sufficient conditions in Theorem 1 admit a weakening: if the matrix $\widetilde{Z}(\zeta)$ is holomorphic in the tube domain T^C (the cone C may also be nonconvex), satisfies inequality (4) in T^C, and the matrix $\widetilde{Z}(iq)$ is real for all $q \in C$, then the matrix $Z(\xi)$ defines a passive operator with respect to the cone C^*.

Theorem 2. *In order that an $N \times N$-matrix $Z(\xi)$ define a passive operator with respect to the cone Γ, it is necessary and sufficient that, for any cone*

$$C' = [q : \ (e_j, q) > 0, \ j = 1, \dots, n] \subset C = \operatorname{int}\Gamma^*$$

it be representable (uniquely) in the form *

$$Z(\xi) = (e_1, D)^2 \dots (e_n, D)^2 Z_{C'}(\xi) + \sum_{j=1}^{n} Z_{C'}^{(j)} D_j \delta(\xi), \tag{5}$$

where $Z_{C'}(\xi)$ is a real continuous matrix of polynomial growth in $\mathbb{R}^n$ with support in the cone $(C')^*$, and such that for any $a \in \mathbb{C}^N$ the generalized function **

$$(e_1, D)^2 \dots (e_n, D)^2([Z_{C'}(\xi) + Z_{C'}^T(-\xi)]a, a)$$

is positive-definite in the sense of Bochner–Schwartz; the matrices $Z_{C'}^{(j)}$, $j = 1, \dots, n$, are real symmetric and such that the matrix

$$\sum_{j=1}^{n} q_j Z_{C'}^{(j)} \geqslant 0$$

for all $q \in C'$.

Remark. For $n = 1$, Theorem 2 was proved by König and Zemanian ([4]).

Corollary. *In order that a function* $f(\zeta)$ *be holomorphic in* T^C, $\operatorname{Im} f(\zeta) \geqslant 0$, $\zeta \in T^C$, *and*

$$\lim \operatorname{Im} f(p + iq) = 0, \qquad q \to 0, \qquad q \in C \text{ in } \mathcal{S}',$$

it is necessary and sufficient that it be representable in the form $f(\zeta) = (a, \zeta)$, *where* $a \in C^*$ *(a theorem of the type of the Liouville and Phragmén–Lindelöf theorems).*

3. Multidimensional dispersion relations

Theorem 3. *In order that an* $N \times N$*-matrix* $Z(\xi)$ *define a passive operator with respect to the cone* Γ, *it is necessary and sufficient that its Fourier transform* $\widetilde{Z}(p)$, *for any cone* $C' \subset C = \operatorname{int} \Gamma^*$ *(see Theorem 2), satisfy the dispersion relation*

$$\widetilde{Z}(p) = \frac{2(-1)^n}{(2\pi)^n} (e_1, p)^2 \dots (e_n, p)^2 (\widetilde{M}_{C'} * K_{C'}) - i \sum_{j=1}^{n} Z_{C'}^{(j)} p_j; \qquad (6)$$

where

$$K_{C'}(p) = \det \|e_{jk}\| \left\{ -i\pi\delta[(e_1, p)] + P\frac{1}{(e_1, p)} \right\} \times \dots$$

$$\dots \times \left\{ -i\pi\delta[(e_n, p)] + P\frac{1}{(e_n, p)} \right\};$$

$\widetilde{M}_{C'}(p) = F[M_{C'}(\xi)]$ *is a Hermitian matrix with the properties: I. The matrix* $M_{C'}(\xi)$ *is real continuous of polynomial growth with support in the cone* $(C')^* \cup (-C')^*$. *II. For any* $a \in \mathbb{C}^N$ *the generalized function*

$$\frac{(-1)^n (e_1, p)^2 \dots (e_n, p)^2}{[1 + (e_1, p)^2] \dots [1 + (e_n, p)^2]} (\widetilde{M}_{C'}(p) a, a)$$

is a nonnegative finite measure on $\mathbb{R}^n$; $Z_{C'}^{(j)}$, $j = 1, \dots, n$, *are real symmetric matrices such that for all* $q \in C'$

* $D = (D_1, \dots, D_n) = (\partial/\partial x_1, \dots, \partial/\partial x_n)$.

** Z^T is the matrix transposed to the matrix Z.

$$\sum_{j=1}^{n} q_j Z_C^{(j)} \geqslant 0.$$

In this case the equality holds

$$M_{C'}(\xi) = \frac{1}{2}\left[Z_{C'}(\xi) + Z_{C'}^T(-\xi)\right]. \tag{7}$$

Remark. For $n = 1$, Theorem 3 was proved by Beltrami and Wohlers [3].

4. Existence of a fundamental solution. A **fundamental solution** of the passive operator $Z*$ is any matrix $A(\xi)$, $A_{kj} \in \mathcal{D}'$, satisfying the matrix convolution equation

$$Z * A = I\delta(\xi). \tag{8}$$

The operator $A*$ is also called the **inverse operator** to the operator $Z*$, and the matrix $A(\xi)$ the (generalized) **admittance** of the system. A passive operator $Z*$ is called **nondegenerate** (**completely nondegenerate**) if $\det Z(\zeta) \neq 0$, $\zeta \in T^C$ (respectively, for every $a \in C^N$, $a \neq 0$, there exists a point $\zeta_0 \in T^C$ such that $\mathrm{Re}(Z(\zeta_0)a, a) > 0$).

In order that the passive operator $Z*$ be completely nondegenerate, it is necessary and sufficient that

$$(Z(\xi)a, a) \neq ig^0(\xi), \qquad a \in C^N, \quad a \neq 0, \quad \mathrm{Im}\, g = 0. \tag{9}$$

The following is the main result.

Theorem 4. *In order that the operator $Z*$, $Z_{kj} \in \mathcal{D}'(\Gamma)$, be a (completely) nondegenerate passive operator with respect to the cone Γ, it is necessary and sufficient that there exist a unique inverse operator $A*$ in the class of (completely) nondegenerate passive operators with respect to Γ.*

5. Examples of passive systems. We first note a sufficient condition for passivity. Let a real $N \times N$ matrix $Z(\xi)$ satisfy the causality condition a), item 1, and the condition

$$\text{b')} \quad \mathrm{Re}\int_{-\Gamma} [((Za, a) * \varphi](x)\,\overline{\varphi}(x)]\,dx \geqslant 0, \qquad \varphi \in \mathcal{D}, \quad a \in C^N. \tag{10}$$

(We assume that the cone Γ satisfies the conditions of item 1 and, moreover, is solid.) Then the matrix $Z(\xi)$ defines a passive operator with respect to the cone Γ, i.e., it satisfies condition b), item 1.

1) If a system of differential equations with constant coefficients is passive, then it is of first order.

2) In order that the differential operator $[(l, D) + c]\delta(\xi)*$ with constant coefficients be passive with respect to the cone Γ, it is necessary and sufficient that $l \in \Gamma$ and $c \geqslant 0$.

3) In order that a system of linear differential equations with real constant coefficients

$$\sum_{k=1}^{n} A_k \frac{\partial u}{\partial x_k} = f, \tag{11}$$

be passive and completely nondegenerate, it is necessary and sufficient that the $N \times N$ matrices A_k, $k = 1, \dots, n$, be symmetric and that there exist such a vector l that

$$\sum_{k=1}^{n} l_k A_k > 0.$$

In this case the passivity of the system (11) holds with respect to the convex hull of the cone

$$[\xi: \ \xi_1 = (A_1 a, a), \dots, \xi_n = (A_{na}, a), \quad a \in R^N].$$

Remark. Systems (11) satisfying the conditions of assertion 3) are the principal parts of systems symmetric in the sense of Friedrichs [13] with constant coefficients.

4) In order that the difference operator

$$B_0 \delta(\xi)_* + \sum_{\nu=1}^{m} B_\nu \delta(\xi - h_\nu)_*,$$

where B_k, $k = 0, 1, \dots, m$, are real $N \times N$ matrices, be passive, it is necessary and sufficient that: I. The smallest closed convex cone Γ containing the points $\{0, h_\nu, \nu = 1, \dots, m\}$ be acute. II. For all $\zeta = p + iq \in T^C$, $C = \operatorname{int}\Gamma^*$, the matrix

$$B_0 + B_0^T + \sum_{\nu=1}^{m} e^{-(q,h_\nu)} \left[\cos(p, h_\nu)(B_\nu + B_\nu^T) - \sin(p, h_\nu) \frac{B_\nu - B_\nu^T}{i} \right] \geqslant 0.$$

In this case passivity holds with respect to the cone Γ.

Correction note. It can be proved that every matrix $Z(\xi)$ defining a passive operator with respect to a solid cone Γ also satisfies condition b'.

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