CYBERNETICS AND CONTROL THEORY

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Abstract

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ON BULGAKOV' S PROBLEM ON THE ACCUMULATION OF PERTURBATIONS

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1. Let us consider the following extremal problem, which is a generalization of B. V. Bulgakov' s problem on the accumulation of perturbations in linear systems (1).

Let the motion of a dynamical system in the phase space X be described by the vector differential equation

$$\dot{x} = F(x, v, t), \tag{1}$$

where $x = \{x^i(t)\}$ is an m-dimensional vector of phase coordinates; $x(0) = \{x^i(0)\}$ is an m-dimensional vector of initial perturbations belonging to the convex closed polyhedron $Y \subset X$; $v = \{v^i(t)\}$ is an r-dimensional vector of constantly acting perturbations belonging to the set V of absolutely continuous bounded vector functions $(|v^i(t)| \leq 1)$, whose derivatives are also bounded $(|\dot{v}^i(t)| \leq s_i)$; $F(x, v, t) = \{F^i(x, v, t)\}$ is an m-dimensional real vector function, defined on a bounded convex set D of the real variables $x^1, \dots, x^m, v^1, \dots, v^r, t$, single-valued and analytic in x, v and piecewise analytic in t.

It is assumed that for any perturbation $w \in W = Y \times V$ there exists a unique solution $x(t)$, all solutions $x(w, t)$ for $w \in W,\ t \in [0; T]$ are bounded in the aggregate, and for any $w \in W,\ t \in [0; T]$ the point $(x(w, t), v(t), t) \in D$.

The maximum deviation of system (1) from the desired state during the time interval $[\tau, T]$ is expressed by means of the functional

$$\varkappa(w) = \max_{\tau \leq t \leq T} [x(t) - p(t)]^* N [x(t) - p(t)], \qquad \tau \in [0; T]. \tag{2}$$

Here the asterisk denotes transposition; $p(t) = \{p^i(t)\}$ is an m-dimensional continuous piecewise analytic vector function characterizing the desired state of system (1); N is a diagonal matrix $(m \times m)$ with ones and zeros on the main diagonal, $\operatorname{tr} N = l,\ 1 \leq l \leq m$.

It is required to determine whether there exist perturbations $w_0 \in W$ (we shall call them worst-case perturbations) at which the functional $\varkappa(w)$ attains its greatest value. If the worst-case perturbations exist, it is required to find all these perturbations and to determine the maximum deviation (2) of system (1) corresponding to them.

It follows from the formulation that the extremal problem (1), (2) is a problem of analyzing the accuracy of dynamical systems of the form (1).

2. We shall show that worst-case perturbations exist in the set W.

Since the set of vector functions $\{\dot{v}(t)\}$ is a weak compact in $L_r^2[0;T]$ (2), the set V will be compact in $C_r[0;T]$. Using this fact, as well as the analyticity of $F(x, v, t)$, the compactness of the set Y, and the assumptions made, one can show that the totality of all solutions $x(t)$ is compact in $C_r[0;T]$. The functional (2) is continuous in $C_r[0;T]$ with respect to $x(t)$. Consequently, there exists a solution $x_0(t)$, and hence also a perturbation w_0, at which the functional (2) attains its greatest value.

The necessary condition for extremality is given by the following

Theorem 1. *In order that the perturbation $w_0 \in W$ be worst, it is necessary that the relation*

$$\max_{w \in W} [x(w_0, t) - p(t)]^* N y(w, t) = [x(w_0, t) - p(t)]^* N y(w_0, t) \quad \text{for } t \in P(w_0), \tag{3}$$

hold, where $y(w, t)$ is the m-dimensional vector solution of the vector differential equation

$$\dot{y} = F_x y + F_v v;$$

$$F_x = \partial F(x_0(t), v_0(t), t)/\partial x, \qquad F_v = \partial F(x_0(t), v_0(t), t)/\partial v, \qquad y(0) \in Y,$$

$$v(t) \in V,$$

and $P(w_0)$ is the set of points $t \in [\tau; T]$ satisfying the condition

$$\max_{\tau \leq \eta \leq T} [x_0(\eta) - p(\eta)]^* N [x_0(\eta) - p(\eta)] = [x_0(t) - p(t)]^* N [x_0(t) - p(t)].$$

The proof is carried out by contradiction, using the derivative of the functional $\varkappa(w)$ at the point w_0 in the direction $\Delta w = w - w_0$, equal to

$$\partial\varkappa(w_0 + \gamma\Delta w)/\partial\gamma\big|_{\gamma=+0} = \max_{t\in P(w_0)} 2[x(w_0,t) - p(t)]^* Ny(\Delta w, t). \qquad (4)$$

3. Taking as the first perturbation w_1 any initial perturbation $x_1(0) \in Y$ and any piecewise-linear perturbation $v_1(t) \in V$, under the condition $\varkappa(w_1) > 0$, we construct a maximizing sequence $\{w_n\}$ by means of the conditional-gradient method (3), modifying it accordingly.

Let

$$w_2 = w_1 + \gamma_1(\widetilde{w}_1 - w_1), \qquad 0 \le \gamma_1 \le 1.$$

It is clear that $w_2 \in W$ if $\widetilde{w}_1 \in W$, by virtue of the convexity of the set W. Since $v_1(t)$ is a piecewise-linear vector function and the vector function $F(x, v, t)$ is single-valued and analytic in x, v and piecewise analytic in t, the solution $x_1(t)$ will be a piecewise-analytic vector function, and the set $P(w_1)$ will consist of a finite number of intervals and a finite number of isolated points. If, instead of intervals, one takes their ε_{k_1}-net, then one obtains the set $P(w_1, \varepsilon_{k_1})$, consisting of a finite number of points. With the aid of the methods presented in (4, 5), we find perturbations $w \in W$ giving the maximum value to the linear functionals

$$[x_1(t) - p(t)]^* Ny(w, t), \qquad t \in P(w_1, \varepsilon_{k_1}).$$

Since F_x, F_v are piecewise-analytic matrices in t, this will require a finite number of steps; moreover, all the constantly acting perturbations $v(t)$ found will be piecewise-linear vector functions. As $\widetilde{w}_1$ we take one of the perturbations found, satisfying the condition

$$\max_{t\in P(w_1,\varepsilon_{k_1})} \max_{w\in W} [x_1(t) - p(t)]^* Ny(w - w_1, t) =$$

$$= [x_1(t_1) - p(t_1)]^* Ny(\widetilde{w}_1 - w_1, t_1), \qquad t_1 \in P(w_1, \varepsilon_{k_1}).$$

If, for any $\varepsilon_{k_1} \to 0$ $(k_1 \to \infty)$,

$$\max_{t\in P(w_1,\varepsilon_{k_1})} \max_{w\in W} [x_1(t) - p(t)]^* Ny(w - w_1, t) = 0,$$

then for any $t \in P(w_1)$

$$\max_{w\in W} [x_1(t) - p(t)]^* Ny(w, t) = [x_1(t) - p(t)]^* Ny(w_1, t),$$

i.e., the perturbation w_1 satisfies the necessary condition for extremality (3).

If, however, for some ε_{k_1}

$$\max_{t\in P(w_1,\varepsilon_{k_1})} \max_{w\in W}[x_1(t)-p(t)]^* Ny(w-w_1,t) = [x_1(t_2)-p(t_1)]^* Ny(\widetilde{w}_1-w_1,t) > 0,$$

then from (4) it follows that for any $\delta \leq [x_1(t_1)-p(t_1)]^* Ny(\widetilde{w}_1-w_1,t_1)$ there exists a $\gamma_1 \in (0;1]$ such that the inequality will be valid:

$$\chi(w_1+\gamma_1(\widetilde{w}_1-w_1))-\chi(w_1) \geq \gamma_1 \left\{ \max_{t\in P(w_1)} 2[x_1(t)-p(t)]^* Ny(\widetilde{w}_1-w_1,t)-\delta \right\} \geq \gamma_1[x_1(t_1)-p(t_1)]^* Ny(\widetilde{w}_1-w_1,t$$

i.e.

$$\chi(w_2) > \chi(w_1).$$

The next stage is repeated analogously, since $w_2 \in W$, $\chi(w_2) > 0$, and $v_2(t)$ is a piecewise-linear vector function, but with the condition $\varepsilon_{k_2} < \varepsilon_{k_1}$. If at the n-th stage the perturbation w_n satisfies the necessary extremality condition (3), then the iterative process terminates. Otherwise we obtain an infinite sequence $\{w_n\}$ and the corresponding infinite strictly increasing sequence of values of the functional (2), $\{\chi_n\}$. It is clear that the sequence $\{\chi_n\}$ has a limit χ_0, which is an upper bound for $\{\chi_n\}$. For the sequence $\{w_n\}$ we introduce the following definition: w_0 is called a limit point of the sequence $\{w_n\}$ if $x_0(0)$ is a limit point of the sequence $\{x_n(0)\}$, and $v_0(t)$ is a limit point of the sequence $\{v_n(0)\}$ in the sense of convergence in the norm in $C_r[0;T]$.

Then the following is valid.

Theorem 2. The sequence $\{w_n\}$ has limit points w_0. All $w_0 \in W$ and satisfy the necessary extremality condition (3).

Remark. Everything said above, with some modifications, also applies to the case when $F(x,v,t)$ is piecewise-continuous in t and has partial derivatives F_x, F_v satisfying the Lipschitz condition in x, v.

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Note: Figure translations are in progress. See original paper for figures.