ON THE INVARIANCE OF DISCRETE SYSTEMS

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Abstract

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Theories of invariance and their applications are the subject of works by G. V. Shchipanov, N. N. Luzin, V. S. Kulebakin, B. N. Petrov, and other authors (for a survey of the problem see ([1])). L. I. Rozonoer in ([2]) devoted attention to the problem of weak invariance and applied it to the investigation of the variational approach. In ([3]) the variational approach is developed in the direction of studying large variations of a functional, and on this basis necessary and at the same time sufficient conditions for weak invariance are obtained and the problem of synthesis of continuous invariant systems is solved. Another method for studying the problem under consideration was used in ([4]). In the present paper we study conditions for weak invariance of discrete systems. The method is based on an approach in many respects analogous to that used in ([3]).

1°. **Formulation of the problem.** Let a discrete perturbed system be described by the system of equations

$$x^{i+1} = f^{i+1}(x^i, u^i), \qquad i = 0, 1, \dots, N-1. \tag{1}$$

Here $x^i \in E_{n_i}$ is the vector of phase coordinates and $u^i \in E_{r_i}$ is the vector of external perturbing actions at the instant i. Let there be given the domains of definition of the right-hand sides of system (1) $G^1, G^2, \dots, G^N,\ G^i \subset E_{n_i} \times E_{r_i}$, and the sequence of domains $A = \{A^0, A^1, \dots, A^N\},\ A^i \subset E_{n_i}$. For each initial condition $x^j \in A^j,\ 0 \le j \le N-1$, a trajectory $x(x^j) = \{x^j, x^{j+1}(x^j), \dots, x^N(x^j)\}$ of system (1) will be considered admissible if there is a set R_{x^j} of perturbations $u = \{u^j, u^{j+1}, \dots, u^{N-1}\}$, for which, for all $i = j, j+1, \dots, N-1$, $\{x^i(x^j), u^i\} \in G^{i+1}$.

Let a quality criterion of system (1) be given,

$$I(x, u) = \Phi(x^N). \tag{2}$$

We shall call system (1) Φ-invariant with respect to u in A if, for any initial conditions $x^j \in A^j$ and any $u \in R_{x^j}$, the value of criterion (2) does not depend on the perturbation u. Our aim will be to determine conditions ensuring the indicated invariance of system (1).

2°. **Conditions of invariance with respect to perturbation.** Suppose that there exists a perturbation $u = \tilde{u}$ belonging simultaneously to all R_{x^j} for $x^j \in A^j$ and $0 \leq j \leq N-1$. We shall call the perturbation $\tilde{u}$ a **reference** perturbation. In A construct the trajectory field $\tilde{x}$ of system (1) corresponding to the perturbation $\tilde{u}$. The field $\tilde{x}$ puts into one-to-one correspondence with each point $x^j \in A^j$ the value of criterion (2), which turns out to be a function of the point x^j. This function

$$V^j(x^j) = \Phi[\tilde{x}^N(x^j)], \qquad 0 \leq j \leq N, \tag{3}$$

will be called the **reference function**. Note that $V^N(x^N) \equiv \Phi(x^N)$.

Let some point $x^k \in A^k$ be given. Compute the difference between the values of criterion (2) for trajectories $\tilde{x}(x^k)$ and $x(x^k) = \{\hat{x}^k(x^k) = x^k, \hat{x}^{k+1}(x^k), \dots, \hat{x}^N(x^k)\}$ starting at x^k, where the first corresponds to the perturbation $\tilde{u}$, and the second to an arbitrary admissible perturbation $\hat{u} = \{\hat{u}^k, \hat{u}^{k+1}, \dots, \hat{u}^{N-1}\}$. Using the constancy of the reference function along

of trajectories $\tilde{x}$, we obtain

$$\Delta I = \Phi[\hat{x}^N(x^k)] - \Phi[\tilde{x}^N(x^k)] = \tilde{V}^N[\hat{x}^N(x^k)] - \tilde{V}^k(x^k) =$$

$$= \sum_{j=k}^{N-1} \left\{ \tilde{V}^{j+1}[\hat{x}^{j+1}(x^k)] - \tilde{V}^j[\hat{x}^j(x^k)] \right\} = \sum_{j=k}^{N-1} \left\{ \tilde{V}^{j+1}[\hat{x}^{j+1}(x^j)] - \tilde{V}^{j+1}[\tilde{x}^{j+1}(\hat{x}^j)] \right\}$$

$$= \sum_{j=k}^{N-1} \left\{ \tilde{V}^{j+1}[f^{j+1}(x^j, \hat{u}^j)] - \tilde{V}^{j+1}[f^{j+1}(x^j, \tilde{u}^j)] \right\}. \tag{4}$$

Directly from formula (4) there follows

Theorem 1 (principle of independence). *In order that system (1) be Φ-invariant with respect to u in A, it is necessary and sufficient that in A the function*

$$\tilde{V}^{j+1}[f^{j+1}(x^j, u^j)], \qquad j = 0, 1, \dots, N-1,$$

corresponding to some reference perturbation $\tilde{u}$, not depend explicitly on u^j.

Formula (4) and Theorem 1 are analogous to the exact formula for the change of a functional and to the theorem on invariance in continuous systems ([3]). However, the role that in the case of continuous systems is played by the Hamiltonian is here performed by the reference function.

Remark. Let us note that an analogous circumstance also occurs in optimization problems for discrete systems. Let the vector u^i in (1) be a control constrained by $u^i \in U^i$. Then, for the case considered here of a free right end of the trajectory of system (1), the following assertion holds, which is an analogue of L. S. Pontryagin' s maximum principle ([5]) in continuous systems:

In order that the control $\bar{u}$ deliver a maximum of the performance criterion (2), it is necessary and sufficient that the maximum conditions be satisfied

$$V^j(x^j) \equiv V^{j+1}[f^{j+1}(x^j, \bar{u}^j)] = \max_{u^j \in U^j} V^{j+1}[f^{j+1}(x^j, u^j)],$$

$$0 \le j \le N-1, \qquad V^N(x^N) \equiv \Phi(x^N).$$

This assertion is essentially a formulation of R. Bellman' s optimality principle ([6]) for the problem under consideration.

Let all the sets R_{x^j} be connected, let the functions $f^{j+1}(x^j, \tilde{u}^j)$ in A^i be continuously differentiable with respect to x^i, and let the functions $f^{i+1}(x^i, u^i)$ in G^i be differentiable with respect to u^i. Construct in A the reference field of vectors $\tilde{p}^j(x^j)$, defining them by the system of relations

$$\tilde{p}^N(x^N) = \operatorname{grad}_{x^N} \Phi(x^N), \qquad \tilde{p}^j_l(x^j) = \left(\tilde{p}^{j+1}[x^{j+1}(x^j)], \frac{\partial}{\partial x^j_l} f^{j+1}(x^j, \tilde{u}^j) \right), \tag{5}$$

$$j = N-1, N-2, \dots, 0, \qquad l = 1, 2, \dots, n^j.$$

Using (3) and (1), we obtain

$$\tilde{p}^j(x^j) = \operatorname{grad}_{x^j} \tilde{V}^j(x^j). \tag{6}$$

It follows from this that, together with the reference function, the reference field $\tilde{p}^j(x^j)$ of system (1), Φ-invariant with respect to u, is invariant with respect to $\tilde{u}$.

Using (6), we obtain for all $j = 0, 1, \dots, N-1$ and $l = 1, 2, \dots, r^j$

$$\frac{\partial}{\partial u^j_l} \tilde{V}^{j+1}[f^{j+1}(x^j, u^j)] = \left(\tilde{p}^{j+1}[f^{j+1}(x^j, u^j)], \frac{\partial}{\partial u^j_l} f^{j+1}(x^j, u^j) \right).$$

With the aid of this equality the invariance conditions can be formulated in terms of the reference field.

Theorem 1a. *In order that system (1) be Φ-invariant with respect to u in A, it is necessary and sufficient that in A, for the reference field corresponding*

for some reference perturbation $\tilde{u}$, the equalities

$$\left(\tilde{p}^{j+1}\left[f^{j+1}(x^j, u^j)\right], \frac{\partial}{\partial u_l^j} f^{j+1}(x^j, u^j)\right) = 0, \quad j = 0, 1, \dots, N-1,$$

$$l = 1, 2, \dots, r^j.$$

3°. Structure of the invariance domain and invariance with respect to initial data. Every trajectory of an invariant system belongs to the manifold

$$\hat{V}_c = \{\hat{V}_c^0, \hat{V}_c^1, \dots, \hat{V}_c^N\}, \quad \hat{V}_c^j = \{x^j \in A^j \mid \hat{V}^j(x^j) = c\},$$

where the constant c is determined by the initial conditions. It is easy to indicate conditions under which, if all $\tilde{p}^j(x^j) \neq 0$, the $\hat{V}_c^j$ are manifolds of dimension less than n^j, and, consequently, the invariant system is not completely controllable. For example, in the case where the vectors x^j have the same dimension $n^j = n$, it is sufficient for this to require that in A^N $\operatorname{grad}_x \Phi(x^N) \neq 0$ and that the Jacobians of the functions $f^{j+1}(x^j, \tilde{u}^j)$ with respect to the variables x^j be nonzero in A^j.

Let a manifold $L = \{L^0, L^1, \dots, L^N\} \subset A$ be given. We shall call system (1) Φ-invariant with respect to the initial data on L if the values of criterion (2) for any of its trajectories beginning at points $x^j \in L^j$ coincide. Obviously, system (1) is Φ-invariant with respect to the initial data on L if and only if $L \subset \hat{V}_c$. Let L^j be smooth manifolds and let $\lambda^j(x^j)$ be an arbitrary tangent vector to L^j at the point x^j. Then, using (6), we obtain the following assertion.

Theorem 2. *In order that system (1) be Φ-invariant with respect to the initial data on L, it is necessary and sufficient that the conditions of Theorem 1 (Theorem 1a) be satisfied and, at every point $x^j \in L^j$,*

$$(\tilde{p}^j(x^j), \lambda^j(x^j)) = 0, \quad j = 0, 1, \dots, N.$$

4°. Synthesis of invariant systems. Let a discrete system be described by the equations

$$x^{i+1} = f^{i+1}(x^i, u^i, v^i), \quad i = 0, 1, \dots, N-1, \tag{7}$$

where v^i is a scalar parameter describing the variable part of the system. We pose the problem of determining such a correcting function

$$v^i = v^i(x^i, u^i), \tag{8}$$

under which the system

$$x^{i+1} = f^{i+1}[x^i, u^i, v^i(x^i, u^i)]$$

is Φ-invariant with respect to u in A. Suppose that, for $u^k = 0$, $k = 0, 1, \dots, N-1$, system (7) functions in the desired manner when $v^k = 0$. Construct in A a reference function $\widehat{V}^j(x^j)$ for system (7), corresponding to $u^k = 0$, $v^k = 0$, $k = 0, 1, \dots, N-1$, and write the equations

$$\widehat{V}^{i+1}\left[f^{i+1}(x^i, u^i, v^i)\right] = \widehat{V}^{i+1}\left[f^{i+1}(x^i, 0, 0)\right], \quad i = 0, 1, \dots, N-1, \tag{9}$$

which determine v^i as a function of x^i and u^i. It follows from Theorem 1 that, in order for function (8) to be correcting, it is necessary and sufficient that it satisfy in A the determining equations (9). As in the case of continuous systems (3), for the synthesis of the correcting function here it is necessary to forecast only the unperturbed motion of the system.

If one assumes that the appropriate continuity and differentiability conditions are satisfied for the right-hand sides of system (7) and function (8), then the determining equations, using Theorem 1a, can be written in the form

$$\left(\tilde{p}^{i+1}\left[f^{i+1}(x^i, u^i, v^i)\right], \left(\frac{\partial}{\partial u_l^i} + \frac{\partial v^i}{\partial u_l^i}\frac{\partial}{\partial v^i}\right) f^{i+1}(x^i, u^i, v^i)\right) = 0,$$

$$i = 0, 1, \dots, N-1, \quad l = 1, 2, \dots, r^i. \tag{10}$$

The reference field $\tilde{p}^j(x^i)$ here corresponds to $u^k = 0$, $v^k = 0$, $k = 0, 1, \dots, N-1$. Equations (10) constitute a system of first-order partial differential equations with respect to the correcting function v^i, whose solution must satisfy the condition $v^i = 0$ when $u^i = 0$.

For the numerical determination of the value of the correcting function compensating a perturbation $\hat{u}^i$ known at the moment i, the following method may be used. Construct in E_{r^i} a piecewise-smooth curve $u^i(s)$, $u^i(s_0) = 0$, $u^i(s_1) = \hat{u}^i$. The function $v^i = v^i(x^i, u^i)$, by virtue of the introduced parametrization, for a fixed value of x^i becomes a function of the parameter s, and our problem consists in finding the number $v^i(x^i, \hat{u}^i) = v^i(s_1)$. We use (9). Differentiating $\nabla^{i+1}[f^{i+1}(x^i, u^i(s), v^i(s))]$ with respect to the parameter s, we obtain

$$\frac{dv^i}{ds} = -\frac{\left(\tilde{p}^{\,i+1}[x^{i+1}(s)], \left[\sum_{l=1}^{r^i} \frac{\partial}{\partial u_l^i} f^{i+1}[x^i, u^i(s), v^i(s)]\frac{du_l^i(s)}{ds}\right]\right)}{\left(\tilde{p}^{\,i+1}[x^{i+1}(s)], \frac{\partial}{\partial v^i} f^{i+1}[x^i, u^i(s), v^i(s)]\right)}.$$

Here $x^{i+1}(s) = f^{i+1}[x^i, u^i(s), v^i(s)]$. To solve this equation it is necessary to compute the values of the vectors of the reference field by the system (5) only for the trajectories $\tilde{x}$ with initial conditions $x^{i+1}(s)$.

The results presented are easily generalized to the case of invariance of the system (1) simultaneously with respect to $m \geq 2$ quality criteria of the form (2). The synthesis of an invariant system in this case is carried out by means of m correcting functions.

5°. **Example.** Consider the problem

$$x_2^{i+1} = x_2^i + \frac{u_1^i u_2^i}{x_1^i} \left[(v^i)^2 - x_1^i x_2^i\right],$$

$$x_2^{i+1} = x_1^i + \frac{v^i x_1^i \left[(u_1^i)^2 x_2^i - (u_2^i)^2 x_1^i\right]}{x_1^i x_2^i + u_1^i u_2^i \left[(v^i)^2 - x_1^i x_2^i\right]}, \qquad i = 0, 1, \dots, N-1;$$

$$\Phi = x_1^N x_2^N.$$

Putting $u_1^k = u_2^k = v^k = 0$, we find $\tilde{x}_1^{i+1} = x_2^i$, $\tilde{x}_2^{i+1} = x_1^i$, $i = 0, 1, \dots, N-1$, and, consequently, $\nabla^j = x_1^j x_2^j$ for all $j = 0, 1, \dots, N$. The determining equations (9) are written in the form

$$(v^i)^2 u_1^i u_2^i + v^i \left[(u_1^i)^2 x_2^i - (u_2^i)^2 x_1^i\right] - u_1^i u_2^i x_1^i x_2^i = 0.$$

Imposing on the correcting function the additional condition of minimality in absolute value, from this we obtain

$$v^i = \frac{u_2^i x_1^i}{u_1^i} \quad \text{when } \left|(u_2^i)^2 x_1^i\right| \ll \left|(u_1^i)^2 x_2^i\right| \text{ and } u_1^i \neq 0,$$

$$v^i = -\frac{u_1^i x_2^i}{u_2^i} \quad \text{when } \left|(u_2^i)^2 x_1^i\right| > \left|(u_1^i)^2 x_2^i\right|, \qquad v^i = 0 \quad \text{when } u_1^i = u_2^i = 0.$$

A correcting function continuous throughout the entire domain of admissible values of the perturbations does not exist in this problem.

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REFERENCES

1. *Technical Cybernetics in the USSR*, "Nauka," Moscow, 1968.

2. L. I. Rozonoer, *Automation and Remote Control*, **24**, Nos. 6 and 7 (1963).

3. V. V. Velichenko, DAN, **184**, No. 2 (1969).

4. M. M. Khrustalev, *Automation and Remote Control*, No. 4 (1968).

5. L. S. Pontryagin, V. G. Boltyanskii et al., *Mathematical Theory of Optimal Processes*, Moscow, 1961.

6. R. Bellman, *Dynamic Programming*, IL, 1960.

Note: Figure translations are in progress. See original paper for figures.