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Abstract

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MATHEMATICS

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ON A STABLE METHOD FOR COMPUTING THE VALUES OF UNBOUNDED OPERATORS

(Presented by Academician A. N. Tikhonov, 10 VII 1968)

1. The stable computation of the values of unbounded operators is one of the most important problems of computational mathematics. Let A be an operator with domain of definition $D_A \subset F$ and range $Q_A \subset U$, where F and U are certain linear normed spaces and $\|A\| = +\infty$. Then there certainly exists a sequence of elements $f_n \in F$, $\|f_n\|_F = 1$, such that $\|Af_n\|_U \to +\infty$. Let $\bar{f} \in D_A$ and $\bar{u} = A\bar{f}$. Put $f_{n,\delta} = \bar{f} + \delta f_n$, where $\delta > 0$ is any arbitrarily small number. Then, obviously, $\|u_{n,\delta} - \bar{u}\|_U \to +\infty$ as $n \to +\infty$, where $u_{n,\delta} = A(\bar{f} + \delta f_n)$, whereas $\|f_{n,\delta} - \bar{f}\|_F = \delta$.

Thus, the problem of computing the values of the operator A in the case under consideration is ill-posed. If one has arbitrary δ-approximations to the element $\bar{f}$, i.e., elements $f_\delta \in F$, $\|f_\delta - \bar{f}\|_F \le \delta$, then it may also happen that the values of the operator A are not even defined on the elements f_δ, i.e., $f_\delta \notin D_A$.

The problem considered below consists in the effective construction of elements $\hat{f}_\delta \in F$ from the given δ-approximations to the element $\bar{f}$, satisfying the following two basic requirements:

1) $\hat{f}_\delta \in D_A$;
2) $\lim\limits_{\delta \to 0} \|\hat{u}_\delta - \bar{u}\|_U = 0$, where $\hat{u}_\delta = A\hat{f}_\delta$.

2. In what follows we shall assume throughout that $U = H$ and F are Hilbert spaces, and that the operator A is linear and closed ([2]), i.e., from the simultaneous fulfillment of the relations

$$f_\nu \in D_A, \qquad \lim_{\nu \to \infty} f_\nu = f, \qquad \lim_{\nu \to \infty} Af_\nu = u$$

it follows that

$$f \in D_A, \qquad u = Af.$$

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Consider the following auxiliary parametric variational problem ($\alpha > 0$ is a parameter) ([1]):

$$\Phi_\alpha[f] = \|f - g\|_F^2 + \alpha \|Af\|_H^2, \qquad f \in D_A \text{ — min}. \tag{1}$$

Theorem 1. *For any given element $g \in F$, problem (1) has a unique solution $f_\alpha \in D_A$.*

Proof. For any f_1 and $f_2 \in D_A$ we have

$$\left\|\frac{f_1 - f_2}{2}\right\|_F^2 + \alpha \left\|\frac{A(f_1 - f_2)}{2}\right\|_H^2 = {}^1/_2\, \Phi_\alpha[f_1] + {}^1/_2\, \Phi_\alpha[f_2] - \Phi_\alpha\left[\frac{f_1 - f_2}{2}\right]. \tag{2}$$

Let

$$m_\alpha = \inf_{f \in D_A} \Phi_\alpha[f].$$

and $f_\nu \in D_A$ is a minimizing sequence for the functional (1):

$$\lim_{\nu \to \infty} \Phi_\alpha[f_\nu] = m_\alpha. \tag{3}$$

Putting $f_1 = f_\nu$, $f_2 = f_{\nu+p}$, where p is an arbitrary natural number, we obtain, by virtue of (3),

$$\left\|\frac{f_\nu - f_{\nu+p}}{2}\right\|_F^2 + \alpha \left\|\frac{Af_\nu - Af_{\nu+p}}{2}\right\|_H^2 = \frac{1}{2}\Phi_\alpha[f_\nu] + \frac{1}{2}\Phi_\alpha[f_{\nu+p}]$$

$$-\Phi_\alpha\left[\frac{f_\nu + f_{\nu+p}}{2}\right] \le \frac{1}{2}\Phi_\alpha[f_\nu] + \frac{1}{2}\Phi_\alpha[f_{\nu+p}] - m_\alpha \to 0,$$

as $\nu \to \infty$, independently of p. But then

$$\lim_{\nu \to \infty} \|f_\nu - f_{\nu+p}\|_F = 0, \qquad \lim_{\nu \to \infty} \|Af_\nu - Af_{\nu+p}\|_H = 0,$$

i.e., the sequences f_ν and Af_ν are fundamental. Let

$$\lim_{\nu \to \infty} f_\nu = f_\alpha, \qquad \lim_{\nu \to \infty} Af_\nu = u_\alpha. \tag{4}$$

Then, by the closedness of the operator A, we have

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$$f_\alpha \in D_A, \qquad Af_\alpha = u_\alpha. \tag{5}$$

From (3), (4), and (5) it follows that

$$m_\alpha = \Phi_\alpha[f_\alpha],$$

i.e., f_α is a solution of problem (1). If $\hat{f}_\alpha$ is some other solution, then, again using (2), we obtain

$$\left\|\frac{\hat{f}_\alpha - f_\alpha}{2}\right\|_F^2 + \alpha \left\|\frac{A\hat{f}_\alpha - Af_\alpha}{2}\right\|_H^2 = m_\alpha - \Phi_\alpha\left[\frac{\hat{f}_\alpha + f_\alpha}{2}\right] \le 0,$$

whence it follows that $\hat{f}_\alpha = f_\alpha$. The theorem is proved.

Corollary. From the theorem just proved it follows that an operator R_α with values in D_A is defined on F:

$$f_\alpha = R_\alpha g, \qquad g \in F.$$

Let us prove that the operator R_α is bounded: $\|R_\alpha\| \le 2$. Indeed,

$$\|f_\alpha - g\|_F^2 + \alpha\|Af_\alpha\|_H^2 \le \|f - g\|_F^2 + \alpha\|Af\|_H^2, \qquad f \in D_A.$$

Putting here $f = 0$, we have

$$\|R_\alpha g\|_F = \|f_\alpha\| \le \|f_\alpha - g\|_F + \|g\|_F \le 2\|g\|_F,$$

which proves our assertion. It can also be shown that the operator R_α is linear.

Next put $f = f_\delta$ in (1) and $f_\alpha^\delta = R_\alpha f_\delta$. The following auxiliary assertion is valid:

Lemma. Let $\delta < \|\operatorname{pr}_{N^\perp} f_\delta\|_F$, where $N^\perp$ is the orthogonal complement of the subspace N of solutions of the homogeneous equation $Af = 0$. Then there exists a unique value of the parameter $\alpha > 0$ such that $\rho(\alpha) = \delta^2$, where $\rho(\alpha) = \|f_\alpha^\delta - f_\delta\|_F^2$.

It can be shown, as in (3), that the values of the function $\rho(\alpha)$ exhaust the interval $[\delta^2, \|\operatorname{pr}_{N^\perp} f_\delta\|_F^2]$, $0 < \alpha \le +\infty$, and $\rho(\alpha)$ increases strictly monotonically. Hence the assertion follows.

Theorem 2. Let $\hat{u}_\delta = A\hat{f}_\delta$, where $\hat{f}_\delta = f_\alpha^\delta$. Then

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$$\lim_{\delta\to 0} \|\hat{u}_\delta - \bar{u}\|_H = 0. \tag{6}$$

i.e., the values of the operator A on the elements $\hat{f}_\delta$ constructed above approximate the sought value of the operator $A\bar{f} = \bar{u}$.

Proof. We have

$$\|f_\alpha^\delta - f_\delta\|_F^2 + \alpha\|Af_\alpha^\delta\|_H^2 \le \|f - f_\delta\|_F^2 + \alpha\|Af\|_H^2, \qquad f \in D_A.$$

Putting here $\alpha = \hat{\alpha}$, $f = \bar{f}$, we obtain

$$\|\hat{f}_\delta - f_\delta\|_F^2 + \hat{\alpha}\|\hat{u}_\delta\|_H^2 = \delta^2 + \hat{\alpha}\|\hat{u}_\delta\|_H^2 \le \|\bar{f} - f_\delta\|_F^2 + \hat{\alpha}\|\bar{u}\|_H^2 < \delta^2 + \hat{\alpha}\|\bar{u}\|_H^2,$$

whence it follows that

$$\|\hat{u}_\delta\|_H \le \|\bar{u}\|_H, \qquad \|\hat{f}_\delta - \bar{f}\|_F \le 2\delta, \tag{7}$$

and, consequently, the family $\hat{u}_\delta = A\hat{f}_\delta$ is weakly compact. Let u_0 be some weak limit point of the family $\{\hat{u}_\delta\}$ and let the subfamily $\{u_{\delta'}\} \subseteq \{\hat{u}_\delta\}$ be such that, for any $u \in H$,

$$\lim_{\delta'\to 0} (u_{\delta'}, u)_H = (u_0, u).$$

Then we have

$$\hat{f}_{\delta'} \to \bar{f}, \qquad \hat{u}_{\delta'} = A\hat{f}_{\delta'} \xrightarrow{\text{w}} u_0.$$

Using the fact of weak closedness of the closed operator A, we obtain

$$A\bar{f} = u_0.$$

On the other hand, $A\bar{f} = \bar{u}$, i.e. the family $\{\hat{u}_\delta\}$ has a unique weak limit point, which is the element $\bar{u}$. Consequently, the element $\bar{u}$ is weakly limiting for the whole family $\{\hat{u}_\delta\}$. Using the known properties of weakly convergent sequences and the first of inequalities (7), we obtain

$$\|\bar{u}\|_H \le \varliminf_{\delta\to 0} \|\hat{u}_\delta\|_H \le \varlimsup_{\delta\to 0} \|\hat{u}_\delta\|_H \le \|\bar{u}\|_H,$$

whence it follows that

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$$\lim_{\delta\to 0}\|\hat{u}_\delta\|_H = \|\bar{u}\|_H.$$

Together with the weak convergence of the family $\{\hat{u}_\delta\}$ to $\bar{u}$, the last relation proves the validity of (6). The theorem is proved.

Theorem 2 completely solves the problem of effective computation of the values of the operator A; here the required algorithm can be written in the form

$$\hat{f}_\delta = R_{\hat{\alpha}} f_\delta, \qquad \hat{u}_\delta = A\hat{f}_\delta.$$

3. As an example of the use of the proposed method, consider the problem of numerical differentiation.

Let $H = F = L_2[a,b]$, and let the operator $Af = d^n f(x)/dx^n$ be the operator of generalized differentiation in the sense of Sobolev. If it is known that $\bar{f}(x) \in D_A$ and δ-approximations to $\bar{f}(x)$ in $L_2[a,b]$ are given:

$$\|\bar{f} - f_\delta\|_{L_2} = \left\{\int_a^b [\bar{f}(x) - f_\delta(x)]^2\, dx\right\}^{1/2} < \delta,$$

then problem (1) reduces to the necessity of solving the Euler equation:

$$\alpha\, d^{2n} f/dx^{2n} + f = f_\delta, \qquad d^{n+i} f(x)/dx^{n+i}\Big|_{x=a,x=b} = 0, \qquad i = 0, 1, \dots, n-1. \tag{8}$$

If the values of the function or of its derivatives at the points a or b are known a priori, then it is necessary to solve problem (8) with the corresponding boundary conditions. The value of the parameter α is found from the equation

$$\int_a^b [f_\alpha^\delta(x) - f_\delta(x)]^2 dx = \delta^2,$$

after which, as an approximation to $\bar{u} = d^n\bar{f}/dx^n$, one may take the function

$$\hat{u}_\delta = d^n f_\alpha^\delta(x)/dx^n.$$

Numerical calculations carried out in application to the problem of differentiation have shown the sufficient effectiveness and simplicity of the proposed method.

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