

**ON A
TWO-DIMENSIONAL
 ζ -FUNCTION
WITH A REAL
PARAMETER**

MATHEMATICS

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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MATHEMATICS

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ON A TWO-DIMENSIONAL ζ -FUNCTION WITH A REAL PARAMETER

(Presented by Academician S. L. Sobolev, 5 V 1968)

§ 1. Let an n -dimensional lattice Γ be given metrically, and let m be a positive number. Then the (n -dimensional) ζ -function of the lattice Γ and of the parameter m is the infinite series, convergent for $m > n/2$,

$$\zeta(\Gamma, m) = \zeta(f, m) = \sum \frac{1}{r^{2m}} = \sum \frac{1}{f^m},$$

where by r are denoted the distances from some point $O \in \Gamma$ to all the remaining points of the lattice Γ , and by $f = \sum a_{ij}x_i x_j$ ($i, j = 1, 2, \dots, n$) is denoted the metric form of some basic parallelotope of the lattice Γ . The second sum, and every similar sum below, is taken over all integral systems (x_1, x_2, \dots, x_n) of the variables of the form f , except for the system $(0, 0, \dots, 0)$. The problem is to find lattices which give, among lattices with the same determinant, a local minimum of the ζ -function for a fixed value of the parameter m . In article ¹ theorems on such minima of a certain special type were proved. In article ² it was proved that the form $x^2 + xy + y^2$, i.e. the lattice constructed on a regular triangle, is optimal in our sense for any natural $m \geq 2$ and $n = 2$.

Fig. 1

In the present article it is proved that, whatever the real number $m \geq 3$, there exist no other lattices giving a local minimum of the two-dimensional ζ -function.

§ 2. Let an arbitrary positive quadratic form $ax^2 + 2bxy + cy^2$ be given; dividing this form by $(a + c)/2$ and making the substitution $t = (c - a)/(c + a)$ and $\alpha = 2b/(c + a)$, we obtain the form $f = (1 - t)x^2 + 2\alpha xy + (1 + t)y^2$. The form $f_1 = f/(1 - \alpha^2 - t^2)^{1/2}$ has determinant $\Delta = 1$. From what has been said it is clear that any positive quadratic form of unit determinant can be represented in this form. Since $1 - \alpha^2 - t^2 > 0$ inside the unit circle of the plane (t, α) ,

we may regard this circle as the parameter space of quadratic forms of unit determinant. It is known that the triangle ABC (see Fig. 1) is equivalent to the Selling reduction domain, and the triangle DOC is the Lagrange reduction domain.

In what follows, if the form f_1 is given in the form

$$f_1 = [(1-t)x^2 + 2\alpha xy + (1+t)y^2]/(1-\alpha^2-t^2)^{1/2} = f/(1-\alpha^2-t^2)^{1/2},$$

then the function $\zeta(f_1, m)$ will be denoted by $\zeta(\alpha, t, m)$.

Now our main theorem can be formulated as follows:

Theorem. The function $\xi(\alpha, t, m)$, for $m \geq 3$, has in the triangle DOC only two critical points: the saddle point D and the point of minimum O .

From our theorem, in particular, it follows trivially that at the point O there is the absolute minimum of the function $\zeta(f_1, m)$ on this triangle, i.e., the optimality of the form $x^2 + xy + y^2$.

§ 3. **Lemma 1.** For $m \geq 3$ and $0 < \alpha < \frac{1}{2}$, the inequality

$$\partial\xi(\alpha, 0, m)/\partial\alpha < 0$$

holds.

Proof. Since

$$\frac{\partial\xi(\alpha, 0, m)}{\partial\alpha} = -(1-\alpha^2)^{m/2-1}m \left[(1-\alpha^2) \sum \frac{2xy}{f^{m+1}} + \alpha \sum \frac{1}{f^m} \right].$$

where $f = x^2 + 2\alpha xy + y^2$, our inequality reduces to the inequality

$$(1-\alpha^2) \sum (2xy/f^{m+1}) + \alpha \sum (1/f^m) > 0,$$

which is obvious for $0 < c_1 < \alpha < c_2 < 0.5$ and for sufficiently large m . We shall now prove this inequality directly for $m \geq 3$ and $0 < \alpha \leq 0.37$. (For $\alpha = 0$, as is easy to see, $\partial\xi(\alpha, 0, m)/\partial\alpha = 0$.)

Since $\sum(1/f^m) > 4$ for any m , we have

$$(1-\alpha^2) \sum (2xy/f^{m+1}) + \alpha \sum (1/f^m) > (1-\alpha^2) \sum (2xy/f^{m+1}) + 4\alpha.$$

Furthermore, since for integral x and y the form $f \geq 1$, we have

$$(1-\alpha^2) \sum (2xy/f^{m+1}) + 4\alpha \geq (1-\alpha^2) \sum (2xy/f^4) + 4\alpha.$$

Using the equality

$$\sum \frac{2xy}{(x^2 + y^2 + 2\alpha xy)^4} = - \sum \left[\frac{4}{1} \alpha \frac{(2xy)^2}{(x^2 + y^2)^5} + \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} \alpha^3 \frac{(2xy)^4}{(x^2 + y^2)^7} + \dots \right]$$

and the inequality $(2xy)^2 \leq (x^2 + y^2)^2$, we obtain

$$(1 - \alpha^2) \sum \frac{2xy}{f^4} + 4\alpha > 4\alpha - (1 - \alpha^2) \sum \frac{(2xy)^2}{(x^2 + y^2)^5} \left[4\alpha + \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} \alpha^3 + \dots \right] =$$

$$= 4\alpha \left[1 - \frac{1 + \alpha^2}{(1 - \alpha^2)^3} \sum \frac{(2xy)^2}{(x^2 + y^2)^5} \right].$$

Substitution in the square brackets, in place of the sum, of its upper estimate 0.5652, and in place of α of a number not exceeding 0.37, gives the bracket a positive value; this proves the first part of the lemma. The assertion of our lemma for $0.37 \leq \alpha < 0.5$ follows from Lemma 2, proved in § 4.

Lemma 2. Suppose that for $m \geq 3$ and some α , where $0.37 \leq \alpha < 0.5$, the equality

$$\partial \xi(\alpha, 0, m) / \partial \alpha = 0$$

is fulfilled; then at this point

$$\partial^2 \xi(\alpha, 0, m) / \partial \alpha^2 > 0.$$

It follows immediately from the lemma that on the interval $0.37 \leq \alpha \leq 0.5$ there can lie only one critical point, but such a point exists—namely the point $\alpha = 0.5$. Consequently, on this interval as well the derivative preserves its sign, which completes the proof of Lemma 1.

§ 4. **Proof of Lemma 2.** Obviously, the sign of the derivative

$$\frac{\partial^2 \xi(\alpha, 0, m)}{\partial \alpha^2} = m(1 - \alpha^2)^{m/2} \left\{ (1 - \alpha^2)^2 (m + 1) \sum \frac{(2xy)^2}{f^{m+2}} + \right.$$

$$\left. + 2m\alpha(1 - \alpha^2) \sum \frac{2xy}{f^{m+1}} + [\alpha^2(m - 1) - 1] \sum \frac{1}{f^m} \right\} = m(1 - \alpha^2)^{m/2} \Phi$$

coincides with the sign of Φ . Using, at the critical point, the relation

$$(1 - \alpha^2) \sum (2xy / f^{m+1}) + \alpha \sum (1 / f^m) = 0,$$

we obtain an expression convenient for us for Φ at this point

$$\Phi = \alpha^2(m + 1) \sum \frac{1}{f^m} + (1 - \alpha^2) \left[2\alpha(m + 1) + \frac{1}{\alpha} \right] \sum \frac{2xy}{f^{m+1}} +$$

$$+ (1 - \alpha^2)^2 (m + 1) \sum \frac{(2xy)^2}{f^{m+2}}.$$

Let us now estimate the expression Φ on the interval $0.37 \leq \alpha \leq 0.5$

$$\begin{aligned} \Phi &> \alpha^2(m+1) \left[4 + \frac{2}{2^m(1+\alpha)^m} + \frac{2}{2^m(1-\alpha)^m} + \frac{4}{(5-4\alpha)^m} \right] \zeta(2m) + \\ &+ (1-\alpha^2) \left[(m+1)2\alpha + \frac{1}{\alpha} \right] \left\{ \left[\frac{4}{2^{m+1}(1+\alpha)^{m+1}} - \frac{4}{2^{m+1}(1-\alpha)^{m+1}} \right. \right. \\ &\left. \left. - \frac{4 \cdot 4}{(5-4\alpha)^{m+1}} \right] \zeta(2m) + B_m \right\} \\ &+ (1-\alpha^2)^2(m+1) \left[\frac{8}{2^{m+2}(1+\alpha)^{m+2}} + \frac{8}{2^{m+2}(1-\alpha)^{m+2}} + \frac{4 \cdot 16}{(5-4\alpha)^{m+2}} \right] \zeta(2m), \end{aligned} \tag{1}$$

(where the term B_m contains all the remaining terms of the sum $\sum 2xy/f^{m+1}$).

Regroup the terms in inequality (1) and strengthen it, taking into account that $B_m > -\pi \cdot 4^{2-m}/3(m-1) - 7^{-m} \cdot 8$ and that for our α the inequalities $(1-\alpha)/(5-4\alpha) > 1/6$ and $5-4\alpha \geq 3$ hold:

$$\begin{aligned} \Phi &> \frac{1}{\alpha}(m\alpha-1) \frac{2\zeta(2m)}{2^m(1-\alpha)^m} + \frac{1}{\alpha}(m\alpha+1) \frac{2\zeta(2m)}{2^m(1+\alpha)^m} + \\ &+ \frac{2}{9\alpha} \{ 24(m+1)\alpha^4 + 13(m+1)\alpha^3 + [12 - 16(m+1)]\alpha^2 + 4\alpha(m+1) - \\ &- 12 \} \frac{2\zeta(2m)}{(5-4\alpha)^m} + \left\{ 4\alpha^2(m+1)\zeta(2m) - \left[\frac{8}{7^m} + \frac{4\pi}{3(m-1)} \frac{1}{4^{m-1}} \right] \right. \\ &\left. \times (1-\alpha^2) \left[(m+1)2\alpha + \frac{1}{\alpha} \right] \right\} \\ &> \frac{1}{\alpha}(m\alpha+1) \frac{2\zeta(2m)}{2^m(1+\alpha)^m} + \frac{2}{9\alpha} \{ [12 - \\ &- 8(m+1)\alpha^2 + 4(m+1)\alpha - 12] \frac{2\zeta(2m)}{(5-4\alpha)^m} \\ &+ \left[4\alpha^2(m+1) - \left(\frac{8}{7^m} + \frac{4\pi}{3(m-1)} \frac{1}{4^{m-1}} \right) (1-\alpha^2) \left[(m+1)2\alpha + \frac{1}{\alpha} \right] \right] \}. \end{aligned}$$

We may write the last inequality, since $\alpha \geq 0.37$ and $m \geq 3$. Further, taking into account that $2^m(1+\alpha)^m < (5-4\alpha)^m$ and that the last term in braces increases with the growth of the parameter m , we have

$$\begin{aligned} \Phi &> \frac{1}{9\alpha} [(8-16m)\alpha^2 + (17m+8)\alpha - 15] \frac{2\zeta(2m)}{(5-4\alpha)^m} + \\ &+ \left\{ 16\alpha^2 - \left(\frac{8}{7^3} + \frac{\pi}{24} \right) (1-\alpha^2) \left(8\alpha + \frac{1}{\alpha} \right) \right\} \geq 0.99 > 0. \end{aligned}$$

which was required to be proved.

§ 5. **Lemma 3.** *On the segment EC , where E is an arbitrary point of the segment DO , the function $\zeta(a, t, m)$ can have no more than one critical point (a minimum of the function $\zeta(a, t, m)$ as a function of one variable).*

Proof. Let the point E have coordinates $(0, \alpha_0)$, and $f_1 \in EC$. The function $\zeta(f_1, m)$ is then written in the form

$$\zeta(a, t, m) = \zeta(f_1, m) = \zeta(\alpha_0(1-t), t, m) = [1-t^2 - \alpha_0^2(1-t)^2] \sum (1/f^m).$$

Taking into account the identity

$$(t + \alpha_0^2 t - \alpha_0^2) f + [1 - t^2 - \alpha_0^2(1-t)^2](-x^2 - 2\alpha_0 xy + y^2) = -(1-t)(x^2 + 2\alpha_0 xy) + [1 + t - 2\alpha_0^2(1-t)]y^2,$$

we write the expression for the derivative $\partial\zeta(f_1, m)/\partial t$ in the form

$$\begin{aligned} \frac{\partial\zeta(f_1, m)}{\partial t} &= -m[1-t^2 - \alpha_0^2(1-t)^2]^{m/2-1} \times \\ &\times \sum \frac{-(1-t)(x^2 + 2\alpha_0 xy) + [1 + t - 2\alpha_0^2(1-t)]y^2}{f^{m+1}}. \end{aligned}$$

Suppose now that at some point t_0 our derivative is equal to zero. Then, using the identity given above, as well as the equality to zero of the infinite sum in the expression for our derivative, we obtain for the second derivative the formula

$$\begin{aligned} \frac{\partial^2\zeta(\alpha_0(1-t_0), t_0, m)}{\partial t^2} &= m[1-t_0^2 - \alpha_0^2(1-t_0)^2]^{m/2-2} \left\{ -\sum \frac{1}{f^m} + \right. \\ &(m+1) \sum [-(1-t_0)(x^2 + 2\alpha_0 xy) + (1+t_0 - 2\alpha_0^2 + 2\alpha_0^2 t_0)y^2]^2 / f^{m+2} \left. \right\} \\ &= m[1-t_0^2 - \alpha_0^2]^{m/2-2} \Psi. \end{aligned}$$

We shall show that the expression Ψ is positive. Indeed:

$$\Psi > (m+1) \left\{ \sum_{y=0} \frac{1}{f^m} + \left[1 - 2\alpha_0^2 \frac{1-t}{1+t} \right]^2 \sum_{x=0} \frac{1}{f^m} \right\} - \sum \frac{1}{f^m} \geq$$

$$\begin{aligned}
 &\geq (m+1) \left(\sum_{y=0} \frac{1}{f^m} + \frac{1}{4} \sum_{x=0} \frac{1}{f^m} \right) - \sum \frac{1}{f^m} > m \sum_{y=0} \frac{1}{f^m} - 2 \sum_{\substack{y>0 \\ xy \neq 1}} \frac{1}{f^m} \geq \\
 &\geq m \sum_{y=0} \frac{1}{f^m} - 2 \sum_{\substack{y>0, x \geq 0 \\ (x,y) \neq (0,1)}} \left[\frac{1}{4}(1-t)x^2 + \frac{3}{4}(1-t)y^2 \right]^{-m} = \\
 &= \frac{2}{(1-t)^m} \left\{ m\zeta(2m) - \left(\frac{4}{3}\right)^m [\zeta(2m) - 1] + \sum_{x>0, y>0} 4^m (x^2 + 3y^2)^{-m} \right\};
 \end{aligned}$$

the last inequality is obtained by shifting, in each row of the lattice, the left half of the row to the right, and the right half to the left, taking into account that the gap between the rows does not exceed $[3(1-t)/4]^{1/2}$.

It remains now to prove that for $m > 3$ the quantity standing in the last braces is positive. But for this it is enough to note that for $m \geq 3$ the following easily verified inequalities hold:

$$\text{A. } \sum_{x>0, y>0} 4^m (x^2 + 3y^2)^{-m} < 2. \quad \text{B. } \left(\frac{4}{3}\right)^m [\zeta(2m) - 1] < 1. \quad \text{C. } m\zeta(2m) > 3.$$

Thus, we have shown that at every critical point the second derivative is positive; consequently, on our segment there can be only minima of the function $\xi(\alpha_0(1-t), t, m)$, but since between two minima there must lie a maximum, this minimum, and thereby the critical point, can be only one.

§ 6. Proof of the theorem. Suppose that in the triangle DOC there is a critical point G of the function $\xi(\alpha, t, m)$, distinct from the points D and O . This point must lie inside the triangle, since, by Lemma 3, the only critical point of the segment OC (the segment DC) is the point O (the point D), and on the segment DO , by Lemma 1, there are no critical points except the points D and O . Draw the straight line CG to its intersection with the segment DO at the point E . The point G is a critical point also for the function $\xi(\alpha_0(1-t), t, m)$, considered on the segment CE ; consequently (Lemma 3), at the point G there is a minimum of this function, and the point G is the only critical point on the segment CE . Therefore, the function $\xi(\alpha, t, m)$ at the point E has a positive derivative in the direction CE . By symmetry, the derivative in the direction of the t -axis at all points of the segment DO , in particular at the point E , is equal to zero. From these propositions it follows that the derivative in the direction of the α -axis at the point E is positive, but this contradicts Lemma 1.

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2. B. N. Delone, N. N. Sandakova, S. S. Ryshkov, DAN, **162**, No. 6 (1965).

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