STATIONARY PROCESSES ON TOPOLOGICAL SEMIFIELDS

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Abstract

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We shall adhere to the definitions and notation of the article ([1]).

1. Let E be a topological semifield; ∇ the topological Boolean algebra of its idempotents; m a measure defined on some maximal confinate $I^* \subset \nabla$, continuous with respect to the topology of the semifield. An element $x \in \overline{K} \cap S$, where $\overline{K}$ is the cone of nonnegative elements of E, and S is the set of all summable elements of E, will be called a probability distribution if $\mu(x) = 1$.

Introduce the notation

$$P = \{x : x \in \overline{K} \cap S,\ \mu(x) = 1\}.$$

Each element $x \in P$ generates a continuous probability measure μ_x, defined on $\nabla^* \subset \nabla$. A random variable in E is specified by a pair $\{\xi, x\}$, where ξ is an element measurable with respect to the measure m ([1]), and $x \in P$.

Let a family of elements of the semifield measurable with respect to the measure m be given in E, $\{\xi_t,\ t \in T\}$.

Definition 1. (see ([2,3])). The quadruple $\{E, \nabla^*, \mu, \xi_t,\ t \in T\}$ is called a random process if the following (consistency) conditions are satisfied:

1) for any permutation $i_1, i_2, \dots, i_n$ of the numbers $1, 2, \dots, n$,

$$\mu_{x_{t_{i_1}, t_{i_2}, \dots, t_{i_n}}} = \mu_{x_{t_1, t_2, \dots, t_n}};$$

2) for any $t_1, t_2, \dots, t_k$ and any $\lambda_1, \lambda_2, \dots, \lambda_k$ $(k < n)$,

$$\mu_{x_{t_1, t_2, \dots, t_n}} \left(\bigwedge_{i=1}^{k} e_{\lambda_i}^{\xi_{t_i}} \right) = \mu_{x_{t_1, t_2, \dots, t_k}} \left(\bigwedge_{i=1}^{k} e_{\lambda_i}^{\xi_{t_i}} \right),$$

where

$$e_\lambda^\xi = \bigvee_{e \in I_\nabla} \{e : \xi e \leq \lambda e\}.$$

Sometimes, for simplicity, instead of $\{E, \nabla^*, \mu, \xi_t,\ t \in T\}$ we shall write ξ_t. Along with the semifield E, we introduce into consideration the complex semifield $\mathcal{E}$, defined by the equality $\mathcal{E} = E + iE$.

Definition 2. Let $\mu(\xi_t x) = \varphi(t)$,

$$\mu\big[(\xi_t - \varphi(t))(\xi_s - \varphi(s))x_{t,s}\big] = v(t,s).$$

If for all $t, s, h,\ \varphi(t+h) = \varphi(t)$ and $v(t+h, s+h) = v(t,s)$, then ξ_t is called a stationary process in the broad sense. If for all $t_1, t_2, \dots, t_n, h$

$$\mu_{x_{t_1+h, t_2+h, \dots, t_n+h}} \left(\bigwedge_{i=1}^{n} e_{\lambda_i}^{\xi_{t_i+h}} \right) = \mu_{x_{t_1, t_2, \dots, t_n}} \left(\bigwedge_{i=1}^{n} e_{\lambda_i}^{\xi_{t_i}} \right),$$

then ξ_t is called a stationary process in the narrow sense.

The following theorem is easily proved.

Theorem 1. If the random process ξ_t is stationary in the narrow sense and $\mu(\xi_0^2 x) < \infty$, then it is stationary also in the broad sense.

If the Gaussian random process ξ_t is stationary in the broad sense, then it is stationary also in the narrow sense.

2. Let us consider the question of the spectral representation of a stationary process in the broad sense.

Definition 3. We shall call a decomposition of the identity of the semifield E a one-parameter family of idempotents e_λ satisfying the conditions:

1) $e_\lambda \cdot e_\gamma = e_\lambda,\ \lambda = \min\{\lambda, \gamma\}$;
2) $e_{\lambda \to 0} = e_\lambda$ (in the sense of the topology of the semifield);
3) $e_{-\infty} = 0,\ e_\infty = 1$, the identity of the semifield.

It is known (4) that any element $z \in E$ has the integral representation

$$z = \int_{-\infty}^{\infty} \lambda \, de_\lambda^z, \tag{1}$$

where e_λ^z is the decomposition of the identity generated by the element z. For the theory of random processes it is important that any process be representable in the form of an integral with respect to some decomposition of the identity.

Let ξ be a measurable element of the semifield and e some idempotent. Denote by $\beta_\xi e$ the inf of the set of those numbers β for which $\xi e \leq \beta e$ on the whole idempotent e. Analogously, $\alpha_\xi e$ means the sup of the set of those numbers α for which $\xi e \geq \alpha e$.

Form the sums

$$s_n = \sum_{i=1}^{n} \alpha_{\xi_t}^{(i)} \cdot (e_{\lambda_{i+1}} - e_{\lambda_i}) \leq \sum_{i=1}^{n} \xi_t \cdot (e_{\lambda_{i+1}} - e_{\lambda_i}) \leq$$

$$\leq \sum_{i=1}^{n} \beta_{\xi_t}^{(i)} \cdot (e_{\lambda_{i+1}} - e_{\lambda_i}) = S_n.$$

Naturally, now in representation (1), instead of λ there will be a function of λ and t. Suppose that the following is fulfilled.

Condition A. As $\max|\lambda_{i+1} - \lambda_i| \to 0$, the sums s_n and S_n have one common limit.

When condition A is fulfilled, we shall write

$$\xi_t = \int_{-\infty}^{\infty} f_{\xi_t}(\lambda)\, de_\lambda \equiv \int_{-\infty}^{\infty} f(\lambda, t)\, de_\lambda. \tag{2}$$

The representation of the process in the form (2) is unique. Indeed, let

$$\xi_t' = \int_{-\infty}^{\infty} f(\lambda, t)\, de_\lambda'$$

be another representation of ξ_t. If $f(\lambda, t) = 0$, then the uniqueness of (2) is clear.

Suppose that $f(\lambda, t) \neq 0$ and $de_\lambda \neq de_\lambda'$. Then

$$\xi_t - \xi_t' = \theta = \int_{-\infty}^{\infty} f(\lambda, t)\, d(e_\lambda - e_\lambda') \neq 0,$$

which leads to a contradiction. Thus, the following has been proved.

Theorem 2. *If condition A is fulfilled, then for any random process $\{E, \nabla^*, \mu, \xi_t, t \in T\}$ and any decomposition of the identity e_λ, there exists a unique function $f(\lambda, t)$ for which*

$$\xi_t = \int_{-\infty}^{\infty} f(\lambda, t)\, de_\lambda.$$

Theorem 3. *A random process $\{E, \nabla^*, \mu, \xi_t, t \in T\}$ is representable in the form (2) if and only if its correlation function is representable in the form*

$$\mu(\xi_t \bar{\xi}_s x) = \int_{-\infty}^{\infty} f(\lambda, t) \bar{f}(\lambda, s) \, dF(\lambda).$$

The proof of the theorem follows from the relations:

1)
$$\mu(((de_\lambda)x)) = \mu(\overline{(de_\lambda)x}) = \mu(((e_{\lambda+d\lambda} - e_\lambda)x)) = F(\lambda+d\lambda) - F(\lambda) = dF(\lambda);$$

2) for any functions $f(\lambda)$ and $g(\lambda)$ that are square-integrable with respect to the measure $dF(\lambda)$,
$$\mu\left(\left(\int_{-\infty}^{\infty} f(\lambda) \, de_\lambda \int_{-\infty}^{\infty} \overline{g(\lambda)} \, de_\lambda\right) x\right) = \int_{-\infty}^{\infty} f(\lambda) \, \overline{g(\lambda)} \, dF(\lambda).$$

If in (2) $f(\lambda, t) = e^{i\lambda t}$, then the random process is stationary in the wide sense.

Corollary 1. *The random process ξ_t is representable in the form*

$$\xi_t = \int_{-\infty}^{\infty} e^{i\lambda t} \, de_\lambda \tag{3}$$

if and only if its correlation function is representable in the form

$$\mu(\xi_t \overline{\xi_s} x) = \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \, dF(\lambda).$$

In view of the uniqueness of the representation (2), every stationary process in the wide sense admits the spectral decomposition (3).

Corollary 2. *The random process ξ_t is stationary in the wide sense if and only if*

$$\xi_t = \int_{-\infty}^{\infty} e^{i\lambda t} \, de_\lambda.$$

Let us note that, with an appropriate choice of the semifield, from (3) we obtain the spectral representations of an ordinary stationary process and of a homogeneous random field.

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REFERENCES

1. T. A. Sarymsakov, Trans. Fourth Prague Conf. on Information Theory, Statistical Decision Functions, Random Processes, Prague, 1967, p. 495.

2. A. N. Kolmogorov, *Basic Concepts of Probability Theory*, Moscow, 1936.

3. T. A. Sarymsakov, V. G. Vinokurov, Dokl. AN UzSSR, No. 5, 3 (1969).

4. M. Ya. Antonovskii, V. G. Boltyanskii, T. A. Sarymsakov, *Metric Spaces over Semifields*, Tashkent, 1962.

Note: Figure translations are in progress. See original paper for figures.