ON SOME PROPERTIES OF \(R\)- AND \(R^c\)-OPERATIONS

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ON SOME PROPERTIES OF R- AND R^c-OPERATIONS

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The article uses the terminology and notation introduced in papers (1–5), and applies the apparatus of D. Kurepa’ s ramified tables.

1. A **ramified table** is a partially ordered system $T = \langle \mathcal{E}, < \rangle$ such that, for every $x \in \mathcal{E}$, the set $\mathcal{P}_x = \{y : y < x\}$ is well ordered. The order type of the set $\mathcal{P}_x$ is called the **order of the element** x. The **rank** $\rho(T)$ of the table T is called $\sup\limits_{x \in \mathcal{E}} \rho(x)$. A **node** of order α is a set $\{x \in \mathcal{E} : \mathcal{P}_x = \mathcal{P}_y\}$ for some $y \in \mathcal{E}$ of order α. The set $T_\alpha = \{x \in \mathcal{E} : \rho(x) = \alpha\}$ is called the **layer** of order α. Elements $a, b \in \mathcal{E}$ are called **comparable** if $a < b$ or $b < a$; otherwise they are called **disjunctive**. A set $U \subset \mathcal{E}$ is called a **section** of the table T if it has a unique common element with each layer T_α. One says that the table T **attains** its rank if it contains a well-ordered subset of type $\rho(T)$.

Remark. A ramified table attains its rank if and only if it admits a monotone section.

Put $(x, \cdot)_T = \{y : y \in \mathcal{E} \text{ and } x < y\}$. A set $U \subset \mathcal{E}$ is called **full** if, for every $x \in U$, one has $\mathcal{P}_x \subset U$. The **completion** $U(T)$ of a set U in the table T is the least full set containing U. Then

$$U(T) = U \cup \left(\bigcup_{x \in U} \mathcal{P}_x \right).$$

Thickening conditions. We shall say that a set U, where $\overline{\overline{U}} = \mathfrak{N}_\tau$, satisfies the conditions: I_τ, if the table $\langle U(T), < \rangle$ attains its rank ω_τ; II_τ, if the table $\langle U(T), < \rangle$ has a node of cardinality $\mathfrak{N}_\tau$.

2. In all that follows we shall consider the set $\mathcal{E}$ countable and $\rho(T) \leq \omega$. The table $T = \langle \mathcal{E}, < \rangle$ will be called **countable**. We note some properties of countable tables.

Theorem 1. Let $U \subset \mathcal{E}$, $T^* = \langle U(T), < \rangle$. If $\rho(T^*) = \omega$ and $F = (x_i)_{i<\omega}$ is a monotone section of the table T^*, then, first,

$$(\forall i)\,[\,\overline{(x_i, \cdot)_{T^*} \cap U}\,] = \mathfrak{N}_0,$$

and, second, either

$$\overline{F \cap U} = \mathfrak{N}_0,$$

or there exists a countable disjunctive subset $B \subset U$ such that

$$(\forall i)\,[\,\overline{(x_i, \cdot)_{T^*} \cap B} = \mathfrak{N}_0\,].$$

Theorem 2. For every countable set $U \subset \mathcal{E}$ in the table $T = \langle \mathcal{E}, < \rangle$, either I_0 or II_0 holds (here $\omega_0 = \omega$).

Denote by I the set of all natural numbers and by W the set of all finite tuples of natural numbers, including the empty tuple $\{\ \}$. Let $T_W = \langle W, < \rangle$, where as the relation $<$ we take the relation of subordination of tuples. Obviously, $\rho(T_W) = \omega$.

3. Let H be a property of sets of chains of a given rigid base N. We shall also denote by H the set of all subsets $H_\xi \subset N$ possessing the property H. Let

   $$HN = \left(\bigcup_{\xi \in H_\xi} \xi\right)'_{H_\xi \in H}.$$

   The following notation is introduced: H_{pN} (p natural), $H_{\mathfrak{N}_0}N$, $H_{\hat{\mathfrak{N}}0}N$, if H_ξ contains, respectively, not fewer than p chains, not fewer than $\mathfrak{N}_0$ chains, more than $\mathfrak{N}_0$ chains of the base N. Let (E_i) be a sequence of sets of the basic space Ξ. The **kernel** of a chain $\xi \in N$ call $\bigcap E_i$. If $x \in \Phi_N(E_i)$, then by M_x we denote the set of all chains $\xi \in N$ to whose kernels the point x belongs. If $I^* \subset I$, then the truncated base N^{I^*} of the operation Φ_N is called $\{\xi \in N : I^* \subset \xi\}$.

4. Consider an $R^c_{\mathfrak{M}^c}$-operation, where $\mathfrak{M}^c = (N^c_{n_1 \dots n_k})_{\{n_1 \dots n_k\} \in W}$ is a table of bases. A. A. Lyapunov (^{4}) showed that, in the case when all bases $N^c_{n_1 \dots n_k}$ are rigid, the $R^c_{\mathfrak{M}^c}$-operation also has a rigid base $\theta^c_{\mathfrak{M}}$. We shall call the $\{n_1 \dots n_k\}$-section of an $R^c_{\mathfrak{M}^c}$-chain ϑ, and denote by $\vartheta_{n_1 \dots n_k}$, the totality of all tuples from the chain ϑ either coinciding with the tuple $\{n_1 \dots n_k\}$ or subordinate to it. Denote by $\Phi_{\vartheta^c_{\mathfrak{M}};\{n_1 \dots n_k\}}$ the δs-operation whose base is $\{\vartheta_{n_1 \dots n_k} : \vartheta \in \theta^c_{\mathfrak{M}}\}$. Obviously, $\Phi_{\vartheta^c_{\mathfrak{M}};\{\}} = \Phi_{\theta^c_{\mathfrak{M}}}$. The totality of tuples $((n_1 \dots n_k n^i_{k+1}))_i$ will be called an R^c-covering of the tuple $\{n_1 \dots n_k\}$, if $\xi = (n^i_{k+1})_i \in N^c_{n_1 \dots n_k}$.

Theorem 3. Every rigid $R^c_{\mathfrak{M}^c}$-chain consists of pairwise non-subordinate tuples.

Theorem 4. If ϑ is a rigid $R^c_{\mathfrak{M}^c}$-chain, and $\vartheta(T_W)$ is its completion in the table T_W, then every node of the table $\langle \vartheta(T_W), < \rangle$ of the form $(\{n_1, \dots, n_k n^i_{k+1}\})_i$ is an R^c-covering of the tuple $\{n_1 \dots n_k\}$.

Theorem 5. No $R^c_{\mathfrak{M}^c}$-chain ϑ satisfies condition I_0, i.e., either the rank of the table $T^* = \langle \vartheta(T_W), < \rangle$ is less than ω, or $\rho(T^*) = \omega$, but the table T^* does not attain it.

Theorem 6. For any tuple $\{n_1 \dots n_k\} \in W$, the operation $\Phi_{\vartheta^c_{\mathfrak{M}};\{n_1 \dots n_k\}}$ is weaker than the operation $\Phi_{\theta^c_{\mathfrak{M}}}$.

Theorem 7. Whatever the tuple $\{n_1 \dots n_k\} \in W$ may be, the operation whose base is an arbitrary truncated base of the operation $\Phi_{\vartheta^c_{\mathfrak{M}};\{n_1 \dots n_k\}}$ is weaker than the operation $\Phi_{\theta^c_{\mathfrak{M}}}$ relative to the class of sets $\mathscr{K} \ni \emptyset$.

A consequence of Theorem 7 and of a theorem of I. Kozlova (^{6}) is

Theorem 8. If the class of sets $\mathscr{K} \ni \emptyset$ and the class $\Phi_{\theta^c_{\mathfrak{M}}}(\mathscr{K})$ are invariant with respect to the operations Σ and Π, then for any tuple $\{n_1 \dots n_k\} \in W$ and any natural p one has

$$\Phi_{H_p(\vartheta^c_{\mathfrak{M}};\{n_1 \dots n_k\})}(\mathscr{K}) \subset \Phi_{\theta^c_{\mathfrak{M}}}(\mathscr{K}).$$

5. Let $x \in \Phi_{\theta^c_{\mathfrak{M}}}(E_{n_1 \dots n_k})$. Denote by η_x the totality of all tuples entering chains that belong to the set M_x.

We introduce the **duality property**. We shall say that a tuple $\{n_1 \dots n_k\} \in \eta_x(T_W)$ has the duality property if from the tuples of the collection η_x one can form at least two chains of the base of the operation $\Phi_{\vartheta^c_{\mathfrak{M}};\{n_1 \dots n_k\}}$. Denote by μ_x the totality of all tuples of the collection $\eta_x(T_W)$ that have the duality property.

Theorem 9. Two distinct $R^c_{\mathfrak{M}^c}$-chains ϑ and ϑ' differ either in that their completions $\vartheta(T_W)$ and $\vartheta'(T_W)$ include subsets μ_x that are distinct from one another, or in that there is a tuple $\{n_1 \dots n_k\} \in \vartheta(T_W) \cap \vartheta'(T_W)$ such that $\vartheta(T_W)$ and $\vartheta'(T_W)$ include R^c-coverings of this tuple that are distinct from one another.

For the $R^c_{\mathfrak{M}^c}$-operation we introduce **condensation conditions**. We shall say that a set of tuples η_x satisfies condition: a_1, if in the table $\langle \eta_x(T_W), < \rangle$ there exists a node $(\{n_1 \dots n_k n^i_{k+1}\})$, from which one can form $> \mathfrak{N}_0$ R^c-coverings of the tuple $\{n_1 \dots n_k\}$; a_2, if in the collection $\eta_x(T_W)$ there are a tuple $\{n_1 \dots n_k\}$ and such an R^c-covering of it $(\{n_1 \dots n_k n^i_{k+1}\})_i$, in which $\mathfrak{N}_0$

tuples have the property of duality; a_3, if in the table $\langle \eta_x(T_W), < \rangle$ there exists a node $\left(\{n_1 \dots n_k n^i_{k+1}\}\right)_i$, from which an $\geqslant \aleph_0$ R^c-covering of the tuple $\{n_1 \dots n_k\}$ can be formed; a_4, if the table $\langle \mu_x, < \rangle$ attains its rank ω.

Theorem 10. For every set η_x one has

$$\overline{\overline{M}}_x \geqslant \aleph_0 \mathbin{\&} \neg a_2 \mathbin{\&} \neg a_3 \Rightarrow a_4.$$

Theorem 11. For every set η_x one has

$$a_1 \vee a_2 \Longleftrightarrow \overline{\overline{M}}_x > \aleph_0.$$

Theorem 12. For every set η_x one has

$$\neg(a_1 \vee a_2) \ \& \ (a_3 \vee a_4) \Longleftrightarrow \overline{\overline{M}}_x = \aleph_0.$$

Theorem 13. For every set η_x one has

$$a_2 \vee a_3 \vee a_4 \Longleftrightarrow \overline{\overline{M}}_x \geqslant \aleph_0.$$

6. Let a δ_s-operation Φ_N and two sequences of sets (E_i), (e_i) be given. Put

$$Q(N; E_i; e_i) = \bigcup_{\xi, \xi'} \left(\bigcap_{i \in \xi} E_i \cap \bigcap_{i \in \xi'} e_i \right),$$

where the union is taken over all pairs (ξ, ξ') of chains such that $\xi \in N$ and ξ' is an infinite subchain of the chain ξ. If N is a rigid base of the operation of countable union, then for any ordinal number $\alpha < \omega_1$ by $\theta^\alpha, \theta^{\alpha c}$ we denote the rigid bases of the operations $R_N^\alpha, R_N^{\alpha c}$, respectively.

Theorem 14. If the class of sets $\mathcal{K} \ni \emptyset$ and for every tuple $\{m_1 \dots m_t\}$ the sets $E_{m_1 \dots m_t}, e_{m_1 \dots m_t}$ belong to the class $\mathcal{K}$, then for any ordinal number $\alpha < \omega_1$ the sets

$$Q_{\{m_1 \dots m_t\}}(\theta^\alpha; E_{m_1 \dots m_t}; e_{m_1 \dots m_t}),$$
$$Q_{\{m_1 \dots m_t\}}(\theta^{\alpha c}; E_{m_1 \dots m_t}; e_{m_1 \dots m_t})$$

belong, respectively, to the classes $\Phi_{\theta^\alpha}(\mathcal{K})$, $\Phi_{\theta^{\alpha c}}(\mathcal{K})$.

7. Let $(E_{m_1} \dots E_{m_t})$ be a table of sets of the class $\mathcal{K}$. We construct the sets

$$\Phi^{\theta c}_{H\aleph_0, \mathfrak{M}}(E_{m_1 \dots m_t}), \quad \Phi^{{\theta_{\mathfrak{M}}}^{c}}_{H\aleph_0, \mathfrak{M}}(E_{m_1 \dots m_t}).$$

Put

$$\mathcal{E}_{n_1 \dots n_k} = \Phi_{\theta_{\mathfrak{M}}^{c;\{n_1 \dots n_k\}}}(E_{m_1 \dots m_t}),$$

$$\mathcal{E}^*_{n_1 \dots n_k} = \Phi_{H\hat{\ }\aleph_0 N^c_{n_1 \dots n_k}}(\mathcal{E}_{n_1 \dots n_k i}), \quad \mathcal{E}^{**}_{n_1 \dots n_k} = \Phi_{H_2(\theta^c_{\mathfrak{M}}; \{n_1, \dots, n_k\})}(E_{m_1 \dots m_t}),$$

$$\mathcal{E}^{***}_{n_1 \dots n_k} = \Phi_{i\, H\aleph_0 N^c_{n_1 \dots n_k}}(\mathcal{E}_{n_1 \dots n_k i}), \quad Y_{n_1 \dots n_k} = Q_i(N^c_{n_1 \dots n_k}; \mathcal{E}_{n_1 \dots n_k i}; \mathcal{E}^{**}_{n_1 \dots n_k i}).$$

Let

$$\theta_{\mathfrak{M}}^{c\{n_1 \dots n_k\}} = \{\vartheta \in \theta^c_{\mathfrak{M}} : \{n_1 \dots n_k\} \in \vartheta\}.$$

Put

$$E^{n_1 \dots n_k}_{m_1 \dots m_t} = \begin{cases} E_{m_1 \dots m_t}, & \text{if } \{m_1 \dots m_t\} \neq \{n_1 \dots n_k\}, \\ \Xi, & \text{if } \{m_1 \dots m_t\} = \{n_1 \dots n_k\}. \end{cases}$$

Let

$$\mathscr{E}^{n_1 \dots n_k} = \Phi_{\theta_{\mathfrak{M}}^{c\{n_1 \dots n_k\}}}(E_{m_1 \dots m_t}), \quad \tilde{\mathscr{E}}^{n_1 \dots n_k} = \Phi_{\theta_{\mathfrak{M}}^{c\{n_1 \dots n_k\}}}(E_{m_1 \dots m_t}^{n_1 \dots n_k}),$$

$$Z^{*}_{n_1 \dots n_k} = \tilde{\mathscr{E}}^{n_1 \dots n_k} \cap \mathscr{E}^{*}_{n_1 \dots n_k}, \quad Z^{**}_{n_1 \dots n_k} = \tilde{\mathscr{E}}^{n_1 \dots n_k} \cap Y_{n_1 \dots n_k}, \quad Z^{***}_{n_1 \dots n_k} = \tilde{\mathscr{E}}^{n_1 \dots n_k} \cap \mathscr{E}^{***}_{n_1 \dots n_k},$$

$$Z^{(1)} = \bigcup_{\{n_1 \dots n_k\} \in W} Z^{*}_{n_1 \dots n_k}, \quad Z^{(2)} = \bigcup_{\{n_1 \dots n_k\} \in W} Z^{**}_{n_1 \dots n_k}, \quad Z^{(3)} = \bigcup_{\{n_1 \dots n_k\} \in W} Z^{***}_{n_1 \dots n_k},$$

$$B_{n_1 \dots n_k} = \mathscr{E}^{n_1 \dots n_{k-1} n_k} \cup \bigcup_{\substack{n'_k \neq n_k \\ \{m_1 \dots m_t\} \in W}} \mathscr{E}^{n_1 \dots n_{k-1} n'_k m_1 \dots m_t}.$$

By $\Phi_{\mathfrak{A}}$ denote the δ_s-operation whose base $\mathfrak{A}$ consists of all chains that are a countable totality of tuples of increasing ranks, each succeeding one being subordinate to the preceding one. It is easy to show that the operation $\Phi_{\mathfrak{A}}$ is weaker than the A-operation. Put

$$Z^{(4)} = \Phi_{\mathfrak{A}}(B_{n_1 \dots n_k}).$$

Theorem 15. $\Phi_{H_{\aleph_0}^{\theta_{\mathfrak{M}}^c}}(E_{m_1 \dots m_t}) = Z^{(1)} \cup Z^{(2)}$.

Theorem 16. $\Phi_{H_{\aleph_0}^{\theta_{\mathfrak{M}}^c}}(E_{m_1 \dots m_t}) = \bigcup\limits_{k=2}^{4} Z^{(k)}$.

Theorem 17. If 1) for any tuple $\{n_1 \dots n_k\} \in W$, for a class of sets $\mathscr{K} \supset \emptyset,\ \Xi$, the conditions are satisfied: a) the operation $\Phi_{N^c_{n_1 \dots n_k}}$ is stronger than the Σ-, Π-operations, b) the operation $\Phi_{H_{\aleph_0} N^c_{n_1 \dots n_k}}$ is weaker than the operation $\Phi_{N^c_{n_1 \dots n_k}}$, c) the $(\theta^c_{\mathfrak{M}};\ N^c_{n_1 \dots n_k})$-type is weaker than the $\theta^c_{\mathfrak{M}}$-type; 2) for an arbitrary class of sets $\mathscr{K}' \supset \emptyset$, the set $Q(N^c_{n_1 \dots n_k};\ E_i;\ e_i)$ belongs to the class $\Phi_{N^c_{n_1 \dots n_k}}(\mathscr{K}')$, whenever the sets $E_i,\ e_i$ belong to the class $\mathscr{K}'$, then

$$\Phi_{H_{\aleph_0}^{\theta_{\mathfrak{M}}^c}}(\mathscr{K}) \subset \Phi_{\theta^c_{\mathfrak{M}}}(\mathscr{K}).$$

Theorem 18. If conditions 1c), 2) of Theorem 17 are preserved and conditions 1a)–1b) are replaced by the conditions: 1a′) the operation $\Phi_{N^c_{n_1 \dots n_k}}$ is stronger than the A-operation, 1b′) the operation $\Phi_{H_{\aleph_0} N^c_{n_1 \dots n_k}}$ is weaker than the operation $\Phi_{N^c_{n_1 \dots n_k}}$, we have:

$$\Phi_{H_{\aleph_0}^{\theta_{\mathfrak{M}}^c}}(\mathscr{K}) \subset \Phi_{\theta^c_{\mathfrak{M}}}(\mathscr{K}).$$

Corollary. For every ordinal number $1 \leq \alpha < \omega_1$, the operations $\Phi_{H_{\aleph_0}^{\theta\alpha c}}$, $\Phi_{H_{\aleph_0}^{\theta^a c}}$ are weaker than the operation $\Phi_{\theta\alpha c}$, and the operation $\Phi_{H_{\aleph_0}^{\theta^a}}$ is weaker than the operation Φ_{θ^a} relative to a class of sets $\mathscr{K} \supset \emptyset,\ \Xi$.

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REFERENCES

1. G. Kurepa, *Ensembles ordonnés et ramifiés*, Publications Mathématiques de l' Université de Belgrade, **4**, 1935, 1.
2. Yu. S. Ochan, UMN, **10**, no. 3, 71 (1955).
3. A. A. Lyapunov, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, **40**, 3 (1953).
4. A. A. Lyapunov, Izv. AN SSSR, ser. matem., **17**, 563 (1953).
5. A. A. Lyapunov, Tr. Mosk. matem. obshch., **6**, 195 (1957).
6. E. I. Kozlova, Izv. AN SSSR, ser. matem., **19**, 125 (1955).

Note: Figure translations are in progress. See original paper for figures.

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