COMPLETELY IRREDUCIBLE REPRESENTATIONS OF CLASS I OF REAL SEMISIMPLE LIE GROUPS

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 519.46

MATHEMATICS

A. I. SHTERN

COMPLETELY IRREDUCIBLE REPRESENTATIONS OF CLASS I OF REAL SEMISIMPLE LIE GROUPS

(Presented by Academician L. S. Pontryagin, January 30, 1969)

1. Let G be a connected semisimple real Lie group with finite center, and let U be its maximal compact subgroup. A weakly continuous representation of the group G by continuous operators in a quasicomplete locally convex linear topological space E will be called completely irreducible if the weakly closed algebra of operators in E containing all the operators of the representation $T(g)$ contains all continuous operators in E. We shall call a completely irreducible representation T a representation of class I (relative to U) if in E there is a vector $\xi \neq 0$ such that $T(u)\xi = \xi$ for all $u \in U$.

Let E' be the space of continuous linear functionals on E, endowed with the topology of uniform convergence on compact convex subsets of E (the Mackey topology); let $[T(g)]'$ be the operator in E' adjoint to $T(g)$. The representation $T' : g \to [T(g^{-1})]'$ will be called the representation adjoint to T.

Completely irreducible representations T, T_1 in the spaces E, F, respectively, are called weakly equivalent if there exists an operator $A : E \to F$ and a formally adjoint operator $A^* : F' \to E'$, carrying a dense invariant set into a dense invariant set, and such that

$$AT(g) = T_1(g)A, \qquad A^*T_1'(g) = T'(g)A^*$$

on the domains of definition of A, A^*, respectively.

2. The topological conditions imposed on the representation space make it possible to pass to a representation of the group algebra X of the group G (X is the collection of finite, infinitely differentiable functions, with the usual linear operations and topology and with multiplication-convolution) by the formula

$$T(f) = \int f(g)T(g)\,dg.$$

The definitions of complete irreducibility and weak equivalence carry over verbatim to representations of the group algebra X. The following proposition was essentially proved in [1] (see also [2]):

Lemma 1. Let T be a completely irreducible representation of G. Then:
1) in E there are no nontrivial closed subspaces invariant with respect to $T(g)$;
2) the adjoint representation is completely irreducible;
3) the representation $g \to T(g)$ is completely irreducible if and only if the corresponding representation of the group algebra is completely irreducible;
4) the representation $g \to T(g)$ is weakly equivalent to the representation $g \to T_1(g)$ if and only if the corresponding representations of the group algebra are weakly equivalent.

3. Let T be a completely irreducible representation of class I in the space E. The formula
$$P = \int_U T(u)\,du$$
defines in E a projector onto the subspace generated by the vectors invariant with respect to U. Consider in X the subalgebra Y,
$$Y = \{f \in X:\ f(ugv) = f(g),\ u, v \in U\}.$$
The subspace PE is invariant with respect to $T(f)$, $f \in Y$. Introduce
$$A_T(f) = T(f)\big|_{PE}.$$
As in [1,2], one proves

Lemma 2. 1) If the representation $g \to T(g)$ of the group G is completely irreducible, then the representation $f \to A_T(f)$ of the algebra Y is completely irreducible.
2) If $g \to T(g)$, $g \to T_1(g)$ are completely irreducible representations

groups G of class I, then the weak equivalence of the representations $f \to A_T(f)$, $f \to A_{T_1}(f)$ implies the weak equivalence of the representations T, T_1.

4. The following construction makes it possible to describe, up to weak equivalence, all completely irreducible representations of the group G of class I with respect to U.

Let $\mathcal{L}_2(U)$ be the Hilbert space of functions $\xi(u)$ on the group U, square-summable with respect to the invariant measure on U. Consider the Iwasawa decomposition of the group G: $G = UAN$. Let M be the centralizer of A in G; $\Gamma = M \cap U$ the centralizer of A in U. Put $\mathcal{L}_2^{(0)} = \{\xi \in \mathcal{L}_2(U) : \xi(u\gamma) = \xi(u), \gamma \in \Gamma\}$. Let $\mathfrak{G}_0$ be the Lie algebra of the group G; $\mathfrak{h}_{R_0}$ the Abelian subalgebra of $\mathfrak{G}_0$ corresponding to the subgroup A; ν a complex-valued linear form on $\mathfrak{h}_{R_0}$. Let $\mathfrak{h}_0$ be a maximal Abelian subalgebra in $\mathfrak{G}_0$ containing $\mathfrak{h}_{R_0}$; $\mathfrak{G}$ the complexification of $\mathfrak{G}_0$, $\mathfrak{h}$ the complexification of $\mathfrak{h}_0$; ρ the half-sum of the positive roots of the algebra $\mathfrak{G}$ with respect to $\mathfrak{h}$; W the Weyl group (the group of permutations of the positive roots). For $u \in U$, $g \in G$ put $g^{-1}u = v(g^{-1}u) \cdot a(g^{-1}u) \cdot n(g^{-1}u)$; $v \in U$, $a \in A$, $n \in N$. Consider in $\mathcal{L}_2^{(0)}(U)$ the representation $\widetilde{T}_\nu$, defined by the formula

$$[\widetilde{T}_\nu(g)\xi](u) = e^{(i\nu-\rho)(\log a(g^{-1}u))}\xi(v(g^{-1}u)). \tag{1}$$

The operators $\widetilde{T}_\nu(g)$ are bounded. The representation $\widetilde{T}_{\bar{\nu}}$ is conjugate to $\widetilde{T}_\nu$.

Let $\widetilde{\mathfrak{M}}_\nu$ be the closure of the cyclic hull of ξ_0, $\xi_0(u) \equiv 1$, in $\mathcal{L}_2^{(0)}(U)$ with respect to $\widetilde{T}_\nu$; analogously, $\widetilde{\mathfrak{M}}_{\bar{\nu}}$ is the closure of the cyclic hull of ξ_0, $\xi_0(u) \equiv 1$, with respect to $\widetilde{T}_{\bar{\nu}}$.

Let $\mathfrak{M}_\nu = \widetilde{\mathfrak{M}}_\nu \cap \widetilde{\mathfrak{M}}_{\bar{\nu}}$, P_ν be the projector in $\mathcal{L}_2^{(0)}(U)$ onto $\mathfrak{M}_\nu$, and T_ν the representation in the space $\mathfrak{M}_\nu$ acting by the formula

$$[T_\nu(g)\xi](u) = P_\nu\{[\widetilde{T}_\nu(g)\xi](u)\}. \tag{2}$$

5. **Lemma 3.** *The representations T_ν are completely irreducible.*

Proof (cf. (1)). Let $\{e_i\}$, $i = 1, 2, \ldots$, be a complete set of irreducible representations of U; Q_{e_i} the projector in $\mathcal{L}_2(G)$ onto the maximal subspace in which the restriction of the regular representation of G to the subgroup U is a multiple of e_i. Let $Q_n = \sum_{i=1}^n Q_{e_i}$. Denote, for $f \in \mathcal{L}_2(G)$: $f^*(g) = \overline{f(g^{-1})}$; $Q_n j Q_n = Q_n[(Q_n f^*)^*]$. Let $Q_{nXQ}n = X_n$. Clearly, $X_n \subset X$. Let E_n be the maximal subspace of $\mathfrak{M}_\nu$ in which the restriction of T_ν to U is a sum of representations multiple of e_i, $i = 1, \ldots, n$. Let P_n be the projector onto E_n in $\mathfrak{M}_\nu$. Since $[\widetilde{T}_\nu(v)\xi](u) = \xi(v^{-1}u)$, $v \in U$, the restriction of $\widetilde{T}_\nu$ to U is a subrepresentation of the regular representation of the group U. Therefore E_n is finite-dimensional. From the construction of T_ν it is evident that $g \to T_\nu(g)$ is irreducible (i.e. contains no nontrivial subrepresentations). Then $f \to T_\nu(f)$, $f \in X$, is irreducible in $\mathfrak{M}_\nu$. Then $f \to T_\nu(f)$, $f \in X_n$, is irreducible in E_n. But $\{T_\nu(f), f \in X_n\}$ is an algebra. By Burnside' s theorem, for any continuous operator A in $\mathfrak{M}_\nu$ there is an $f \in X_n$ such that $P_{nT_\nu}(f)P_n = P_{nAP}n$ on E_n, while on $E_n^\perp$ the latter equality is obvious; since $P_{nAP}n \to A$, the lemma is proved.

6. **Theorem.** 1) *Every completely irreducible representation of the group G of class I with respect to U is weakly equivalent to one of the representations T_ν.*

2) *The representations T_ν, T_λ are weakly equivalent if and only if ν and λ lie on the same orbit with respect to the Weyl group.*

Proof. Let T be a completely irreducible representation of the group G of class I with respect to U in the space E. Pass to the representation $f \to A_T(f)$ of the subalgebra $Y \subset X$ in the space PE. The subalgebra Y is commutative (see

([3])), hence any of its completely irreducible representations is one-dimensional (see ([1])), i.e. $A_T : f \to A_T(f)$ is a homomorphism

Y in C. According to the definition of $A_T(f)$, we have

$$A_T(f)\xi = \left[\int f(g)\eta(T(g)\xi)\, dg\right] \cdot \xi$$

for $\xi \in PE$, $\eta \in PE'$ such that $\eta(\xi) = 1$. The function $\Phi_T(g) = \eta(T(g)\xi)$ is continuous and

$$\eta(T(ugv)\xi) = \eta(T(u)T(g)T(v)\xi) = \eta(T(g)\xi),$$

i.e. $\Phi_T(g)$ is bi-invariant with respect to U. By (3) (see also (6), p. 445, Lemma 4.2), $\Phi_T(g)$ is an eigenfunction for every differential operator on G invariant with respect to left translations from G and right translations from U. According to (4), $\Phi_T(g)$ is equal to

$$\Phi_\nu(g) = \int_U e^{(i\nu-\rho)(\log a(g^{-1}u))}\, du$$

for some complex-valued linear form ν on $\mathfrak{h}_{\mathfrak{p}0}$. But

$$\int f(g)\Phi_T(g)\, dg = \int f(g)\Phi_\nu(g)\, dg = \int f(g)(T_\nu(g)\xi_0, \xi_0)\, dg = (T_\nu(f)\xi_0, \xi_0),$$

where $\xi_0(u) \equiv 1$, i.e. the representation T is weakly equivalent to T_ν by Lemma 2. The second part of the theorem follows from Lemma 2 and the relation for the functions Φ_ν ((5), see also (6)): $\Phi_\nu \equiv \Phi_\lambda$ if and only if ν and λ lie on one orbit with respect to the Weyl group.

7. In (7) a number of results are reported on representations X^λ, $\widetilde{X}^\lambda$ of the enveloping algebra $\mathfrak{U}$ of the Lie algebra $\mathfrak{G}$, corresponding to $\widetilde{T}_\lambda$ and $\widetilde{T}_\lambda|_{\mathfrak{M}_\lambda}$ (X^λ, $\widetilde{X}^\lambda$ act in the space of U-finite vectors in the spaces of the representations $\widetilde{T}_\lambda$, $\widetilde{T}_\lambda|_{\mathfrak{M}_\lambda}$, respectively). In particular, it is indicated that any algebraically irreducible representation of $\mathfrak{U}$ in a linear space E with a one-dimensional subspace annihilated by U is algebraically equivalent to one of the representations $\widetilde{X}^\lambda$; moreover, a Weyl chamber C is singled out such that, for $\lambda \in C$, the representation $\widetilde{X}^\lambda$ is algebraically irreducible, and necessary and sufficient conditions on λ are given under which $X^\lambda = \widetilde{X}^\lambda$.

8. Let G be the group of k-rational points of a simple, simply connected, connected, quasisplit algebraic group over a finite extension k of the field of p-adic numbers. Let U be a maximal compact subgroup of the group G; let S be a maximal torus in G split over k; let H be the centralizer of S, and suppose that the Iwasawa and Cartan decompositions of the group G (see (8))

$$G = UHU = UHN$$

(N is a vector k-unipotent subgroup of G, normalized by H) satisfy Condition II in (9). Let u be the group of units of the field k;

$$H^u = \{h \in H \mid \chi(h) \in u \text{ for all } k\text{-morphisms } \chi \text{ of the group } H \text{ into } k^*\}.$$

Let $H^u \subset U$.

Let a be a mapping of H into $\mathbb{C}^*$; let $\mathcal{H}^\alpha$ be the Hilbert space of functions on G:

$$f(ghn) = \alpha(h)f(g), \qquad g \in G, \quad h \in H, \quad n \in N;$$

$$\|f\|^2 = \int_U |f(v)|^2\, dv,$$

where dv is Haar measure on U; the representation T^α is defined by the formula

$$[T^\alpha(g_0)f](g) = f(g_0^{-1}g).$$

T^α is of class I with respect to U (see item 1) if and only if $\alpha(H^u) = 1$. Define T^α as in (2), item 4.

Using the arguments of items 2-6 and Theorem 2 in (9), we obtain that any completely irreducible representation of G of class I with respect to U is weakly equivalent to one of the representations T^α (one can also indicate a group W acting in $\mathrm{Hom}(H, \mathbb{C}^*)$, to whose orbits the different spherical functions correspond one-to-one (see (9)), and hence also the inequivalent representations).

In conclusion, the author expresses gratitude to Prof. M. A. Naimark for his attention to this work.

Moscow State University
named after M. V. Lomonosov

Received
27 XII 1969

References

1. M. A. Naimark, *Linear representations of the Lorentz group*, Moscow, 1958.

2. D. P. Zhelobenko, M. A. Naimark, *Dokl. Akad. Nauk SSSR*, **171**, No. 1, 25 (1966).

3. I. M. Gelfand, *Dokl. Akad. Nauk SSSR*, **70**, No. 1 (1950).

4. Harish-Chandra, *Trans. Am. Math. Soc.*, **76**, 26 (1954).

5. Harish-Chandra, *Am. J. Math.*, **80**, 241 (1958).

6. S. Helgason, *Differential Geometry and Symmetric Spaces*, Moscow, 1964.

7. B. Konstant, *On the Existence and Irreducibility of Certain Series of Representations*. Preprint.

8. F. Bruhat, J. Tits, *Collection of translations. Mathematics*, **12**, 5, 19 (1968).

9. I. Satake, *Publ. Math. Inst. Haut. Et. Sci.*, No. 18, 5 (1963).

Note: Figure translations are in progress. See original paper for figures.