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Abstract

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MATHEMATICS

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ON THE SPECTRUM OF SELF-ADJOINT EXTENSIONS OF A SYMMETRIC SEMI-BOUNDED OPERATOR

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1. Consider in a Hilbert space $\mathcal{H}$ a closed symmetric operator S, defined on an everywhere dense set $D(S)$. We shall assume that the operator S is semibounded from below. Without loss of generality one may suppose that

$$(Sf, f) \geqslant (f, f), \qquad f \in D(S). \tag{1}$$

From condition (1) it follows that the deficiency indices of the operator S are equal to each other and coincide with the dimension of the subspace U of solutions of the equation

$$S^* u = 0. \tag{2}$$

In formula (2), S^* is the operator adjoint to the operator S.
In what follows we shall assume that the condition

$$\dim U = \infty \tag{3}$$

is fulfilled.

As K. O. Friedrichs first showed, there exists a self-adjoint extension S_μ of the operator S whose lower bound is equal to one. Consequently, there exists the inverse operator S_μ^{-1}, defined on all of $\mathcal{H}$. We shall assume that the operator S_μ^{-1} is completely continuous, and hence the spectrum of the operator S_μ is discrete.

Denote by $\lambda_n(S_\mu)$ $(n = 1, 2, ...)$ the sequence of eigenvalues of the operator S_μ, numbered in increasing order. Suppose further that, as $n \to \infty$,

$$\lim \frac{\lambda_n(S_\mu)}{n^\alpha} = a, \qquad a > 0, \quad \alpha > 0. \tag{4}$$

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For brevity we shall agree to write condition (4) in the form

$$\lambda_n(S_\mu) \sim a n^\alpha \qquad (n \to \infty).$$

Under the assumptions made, the following is true.

Theorem 1. *Whatever the numbers $0 < \beta < \alpha$ and $b(\beta) > 0$ may be, there always exists at least one nonnegative self-adjoint extension $\widetilde{S}$ of the operator S with discrete spectrum, whose eigenvalues $\lambda_n(\widetilde{S})$ satisfy the condition*

$$\lambda_n(\widetilde{S}) \sim b n^\beta \qquad (n \to \infty). \tag{5}$$

In the proof of the theorem, certain theorems of M. I. Vishik ([1]), M. Sh. Birman ([2]), and the asymptotic theorem of Fan Zhou ([3]) are used in an essential way.

2. Let us indicate an application of the results obtained to the study of the spectrum of self-adjoint extensions of symmetric operators generated by differential expressions.

Let D be the interior of the unit disk bounded by the circle Γ. Denote by S_0 the operator in $L^2(D)$ generated by the differen-

differential expression $-(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ on the domain of definition $D(S_0)$, consisting of all functions twice continuously differentiable in $D + \Gamma$, each of which is equal to zero in some contour strip of the domain D. We shall further denote by S the closure of the operator S_0 in the metric of $L^2(D)$, and by $D(S)$ its domain of definition. It is known that the operator S is symmetric and semibounded below.

M. Sh. Birman ([2]) showed that the domain of definition $D(\widetilde{S})$ of a self-adjoint extension $\widetilde{S}$, for which $\widetilde{S}^{-1}$ is a completely continuous operator, is described by the relation

$$D(\widetilde{S}) = D(S) + (S_\mu^{-1} + B)U. \tag{5'}$$

In this formula B is a self-adjoint completely continuous operator for which the subspace U is invariant, and on $\mathcal{H} \ominus U$ the operator $B = 0$.

We shall assume henceforth that the operator B maps smooth harmonic functions into smooth ones. Under this assumption it follows from formula $(5')$ that $\widetilde{S}$ is the closure of its restriction to sufficiently smooth functions belonging to $D(\widetilde{S})$. In what follows, without stating this separately, when speaking of the domain of definition $D(\widetilde{S})$ we shall mean precisely the smooth functions belonging to $D(\widetilde{S})$. The subspace U coincides with the totality of all harmonic functions $u(x, y) \in L^2(D)$. Consequently, condition (3) is fulfilled. In addition, the operator S_μ^{-1} is completely continuous. Thus all the hypotheses of Theorem

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1 are satisfied. Following M. I. Vishik ([1]), we define the boundary operators γ_1 and γ_2 as follows:

$$\gamma_1 f = f|_\Gamma; \qquad \gamma_2 f = \frac{\partial f}{\partial n}\Big|_\Gamma - \frac{\partial \bar{f}}{\partial n}\Big|_\Gamma; \qquad f \in D(\widetilde{S}). \tag{6}$$

In formulas (6) the function $\bar{f}$ is defined as the solution of the Dirichlet problem

$$\Delta \bar{f} = 0; \qquad \bar{f}|_\Gamma = \gamma_1 f.$$

Define in $L^2(\Gamma)_{2\pi}$ the operator A by the formula

$$Af = \int_0^{2\pi} G(s,\eta) f(\eta)\, d\eta. \tag{7}$$

In formula (7) the function $G(s,\eta)$ is defined by the relation

$$G(s,\eta) = \iint_D K(\bar{x}, s) K(\bar{x}, \eta)\, dx\, dy, \qquad \bar{x} = (x, y);$$

$K(\bar{x}, s)$ is the Poisson kernel for the disk. It is easy to verify that the operator A so defined is self-adjoint, positive, and completely continuous. It is not difficult to show that there exists an operator

$$C = (\gamma_2 S_\mu^{-1} \gamma_1^{-1})^{-1}, \tag{8}$$

and, moreover, if we denote $f_1(s) = \gamma_1 f$, $f_2(s) = \dfrac{\partial f}{\partial n}\Big|_\Gamma$, then

$$C\gamma_2 f = H_1 f_1 + H_2 f_2. \tag{9}$$

The operators H_1 and H_2 in the case of the disk can be computed explicitly. We note that $H_1 = A^{-1} H_0$, $H_2 = A^{-1}$.

Theorem 2. Let T be an arbitrary self-adjoint nonnegative completely continuous operator in $L^2(\Gamma)$. In order that the spectrum of the self-adjoint extension $\widetilde{S}$ of the operator S be discrete, it is sufficient that $f(\bar{x}) \in D(\widetilde{S})$ satisfy the relation

$$(A^{1/2} - TA^{1/2} H_1) f_1 = TA^{1/2} H_2 f_2. \tag{10}$$

If $\lambda_n(T) \sim 1/bn^\beta$, then

$$\lambda_n(\widetilde{S}) \sim bn^\beta \qquad (0 < \beta < \alpha). \tag{11}$$

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3. Suppose now it is known that some extension $\widetilde{S}$ of the operator S is determined by boundary operators L and M in $L^2(\Gamma)$, i.e., by the relation

$$L_1 f_1 = M f_2. \tag{12}$$

The question is whether $\widetilde{S}$ will be a self-adjoint extension of S, and whether the spectrum of $\widetilde{S}$ is discrete.

Theorem 3. If the operator

$$T = A^{1/2}(MH_0 + L)^{-1} M A^{1/2} \tag{13}$$

is self-adjoint and completely continuous in $L^2(\Gamma)$, then the extension $\widetilde{S}$ of the operator S, determined by the boundary conditions (12), will be self-adjoint in $L^2(D)$, and the spectrum of this extension is discrete.

Thus, the question of the self-adjointness of the boundary-value problem (12) is reduced to the question of the self-adjointness in $L^2(\Gamma)$ of the operator $(MH_0 + L)^{-1}M$. It is not difficult to indicate sufficient conditions for the self-adjointness of this operator.

4. We shall call D-extensions of the operator S those self-adjoint extensions $\widetilde{S}$ for which

$$(\widetilde{S}f, f) = \iint_D \left(\left| \frac{\partial f}{\partial x} \right|^2 + \left| \frac{\partial f}{\partial y} \right|^2 \right) dx\, dy. \tag{14}$$

Note that the self-adjoint extension A_0 of the operator S, characterized by the boundary condition $f|_\Gamma = 0$, is a D-extension.

Theorem 4. The spectrum of any D-extension is discrete and

$$\lambda_n(D) \sim \lambda_n(A_0) \qquad (n \to \infty).$$

5. An obvious generalization of the results obtained is possible for arbitrary domains D and elliptic differential operators.

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Note: Figure translations are in progress. See original paper for figures.

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