ELASTIC-PLASTIC STRESS DISTRIBUTION IN A PLANE WITH A HOLE

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Fig. 1

Figure 1: Fig. 1

Abstract

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THEORY OF ELASTICITY

B. D. Annin

ELASTIC-PLASTIC STRESS DISTRIBUTION IN A PLANE WITH A HOLE

(Presented by Academician Yu. N. Rabotnov, 14 V 1968)

Let the plane xy with a hole—a simply connected domain G^+, bounded by a strictly convex sufficiently smooth contour Γ—be under the action of balanced systems of forces applied to the contour of the hole and at infinity. Denote by G^- the infinite domain bounded by Γ; by σ_x, σ_y, τ the components of the stress tensor in the coordinate system x, y; and by $R(\sigma_x, \sigma_y, \tau) = k$ the plasticity condition. We shall assume that at infinity $\sigma_x = \alpha_1 = \text{const}$, $\sigma_y = \beta_1 = \text{const}$, $\tau = 0$. Suppose that there is a domain in which the material is in a plastic state. We denote the unknown boundary separating the elastic and plastic domains (Fig. 1) by L, and assume that L is a simple Jordan curve of class $C^{2,\lambda}$, $0 < \lambda \leq 1$*. Let D^- and D^+ be, respectively, the finite and infinite domains bounded by L. Assume that $G^+ \subset D^+$.

We shall attach the index p to the components of the stress tensor in the plastic domain $D^+ - G^+$, and the index e in the elastic domain D^-. On L the limiting equalities $\sigma_x^p = \sigma_x^e$, $\sigma_y^p = \sigma_y^e$, $\tau^p = \tau^e$ hold.

Let the stresses in the plastic domain be found. Denote

$$h(x, y) = 4^{-1}\,[\sigma_x^p(x, y) + \sigma_y^p(x, y)],$$

$$f(x, y) = 2^{-1}\,[\sigma_y^p(x, y) - \sigma_x^p(x, y) + 2i\tau^p(x, y)],$$

$$i^2 = -1.$$

Fig. 1

We shall assume that the functions $h(x,y)$, $f(x,y)$ are defined and three times continuously differentiable for all points of the domain G^-. Suppose that the function $R_1(x,y) \equiv R(\sigma_x^e(x,y), \sigma_y^e(x,y), \tau^e(x,y))$ attains its maximum value on L; the function $[\sigma_x^e(x,y) + \sigma_y^e(x,y)]$ belongs$^{(1)}$ to the class $A^{2,\lambda}(D^- + L)$, and the function $[\sigma_y^e(x,y) - \sigma_x^e(x,y) + 2i\tau^e(x,y)]$ to the class $A^{1,\lambda}(D^- + L)$. Denote the unit circle $|\zeta| = 1$ of the ζ-plane by C, and the infinite domain outside C by K^-.

Let the function

$$z = x + iy = \omega(\zeta) = c_0\zeta + c_1/\zeta + c_2/\zeta^2 + \cdots, \quad c_0 > 0,$$

map conformally the domain K^- onto D^-. Denote

$$H(\omega(\zeta), \overline{\omega(\zeta)}) \equiv h[(\omega(\zeta) + \overline{\omega(\zeta)})/2,\ (\omega(\zeta) - \overline{\omega(\zeta)})/2i],$$

$$F(\omega(\zeta), \overline{\omega(\zeta)}) \equiv f[(\omega(\zeta) + \overline{\omega(\zeta)})/2,\ (\omega(\zeta) - \overline{\omega(\zeta)})/2i].$$

* The second derivative of the radius vector with respect to arc length is Hölder-continuous with exponent λ.

If the Kolosov-Muskhelishvili representation is used, then the problem of determining the elastic region and the stresses in the elastic region (problem 1) is formulated as follows.

Problem 1. Find the functions $\omega(\zeta)$, $\Phi(\zeta)$, $\Psi(\zeta)$ of the complex variable ζ, analytic in K^-, $|\zeta| > 1$, where $\Psi(\zeta)$ belongs to the class $A^{1,\lambda}(K^- + C)$, and the functions $\omega(\zeta)$, $\Phi(\zeta)$ to the class $A^{2,\lambda}(K^- + C)$, under the following conditions:

a) on C *

$$\operatorname{Re}\Phi(\tau) = H(\omega(\tau), \overline{\omega(\tau)}), \quad \tau \in C; \tag{1}$$

$$\overline{\omega(\tau)}[\Phi'(\tau)]^-/\omega'(\tau) + [\Psi(\tau)]^- = F(\omega(\tau), \overline{\omega(\tau)}); \tag{2}$$

b) at infinity the following representation holds

$$\Phi(\zeta) = \alpha + a_2/\zeta^2 + a_3/\zeta^3 + \cdots,$$

$$\Psi(\zeta) = \beta + b_2/\zeta^2 + b_3/\zeta^3 + \cdots; \tag{3}$$

c) the function $z = \omega(\zeta)$, $\zeta \in K^-$, maps K^- onto some simply connected domain D^+ with boundary L, and D^+ contains the prescribed domain G^+.

Remark 1. The quantities α, β are parameters; other parameters may enter the functions $H(\omega(\tau), \overline{\omega(\tau)})$, $F(\omega(\tau), \overline{\omega(\tau)})$.

Problem 1 was considered in ($^{2-6}$).

Suppose that the solution of problem 1 exists, and let us find the equation satisfied by the function $\omega(\tau)$, $\tau \in C$. From (1) we find ($\zeta \in K^-$, $\tau \in C$)

$$\Phi(\zeta) = \frac{1}{\pi i} \int_C \frac{H(\omega(\tau), \overline{\omega(\tau)})}{\zeta - \tau} \, d\tau + \alpha, \qquad \frac{1}{2\pi i} \int_C \frac{H(\omega(\tau), \overline{\omega(\tau)})}{\tau} \, d\tau = \alpha. \tag{4}$$

Here and below, in integration the contour C is traversed so that the domain K^- remains on the right.

From (3) and (4) we obtain ($\zeta \in K^-, \tau \in C$)

$$\Phi'(\zeta) = \frac{1}{\pi i} \int_C \frac{\mu(\tau)}{\zeta - \tau} \, d\tau, \qquad \int_C \frac{\mu(\tau)}{\tau} \, d\tau = 0, \tag{5}$$

where **

$$\mu(\tau) = A(\omega(\tau), \overline{\omega(\tau)})\omega'(\tau) - \tau^{-2} \cdot \overline{A(\omega(\tau), \overline{\omega(\tau)}) \cdot \omega'(\tau)}, \tag{6}$$

$$A(\omega(\tau), \overline{\omega(\tau)}) = \frac{1}{2} \Big[h_x[(\omega(\tau) + \overline{\omega(\tau)})/2] - i h_y[(\omega(\tau) + \overline{\omega(\tau)})/2,$$

$$(\omega(\tau) - \overline{\omega(\tau)})/2i] \Big] .$$

Taking into account (^{7,8}), the condition satisfied by the boundary value $[\Psi(\tau)]^-$, $\tau \in C$, and using the first of conditions (5), we obtain the following nonlinear singular integro-differential equation for the boundary values of the analytic function $\omega(\zeta)$:

$$t\Lambda^+(t) - 2\gamma t = -\frac{2}{\pi} \int_C \frac{\overline{\omega(t)} - \overline{\omega(\tau)}}{\overline{t} - \overline{\tau}} \, \mu(\tau) \, d\theta,$$

$$\Lambda^+(t) = \omega'(t) F(\omega(t), \overline{\omega(t)}) + \frac{1}{\pi i} \int_C \frac{\omega'(\tau) F(\omega(\tau), \overline{\omega(\tau)})}{t - \tau} \, d\tau, \tag{7}$$

$$\tau, t \in C, \qquad \gamma = c_0 \beta, \qquad \tau = e^{i\theta}.$$

* $\Phi'(\zeta) \equiv d\Phi(\zeta)/d\zeta$; $\omega'(\zeta) \equiv d\omega(\zeta)/d\zeta$; Re is the real part; the bar over a function denotes the operation of taking the conjugate value; $[\Psi(\tau)]^-$ is the limiting value on C from the domain K^- of the function $\Psi(\zeta)$; $\alpha = (\alpha_1 + \beta_1)/4$; $\beta = (\beta_1 - \alpha_1)/2$.

** $h_x \equiv \partial h(x,y)/\partial x$, $h_y \equiv \partial h(x,y)/\partial y$.

Let $\omega(\tau)$, $\tau \in C$, be a solution of equation (7) having a second derivative continuous in the Hölder sense with exponent λ, and let $\omega(\tau)$ be the boundary value of a function $z = \omega(\zeta)$, analytic in K^-, which maps K^- onto some domain D^- with boundary L, where D^+ contains the prescribed domain G^+. Then, using (2), (3), (4), we determine functions $\Phi(\xi)$, $\Psi(\xi)$, analytic in K^-, which together with $\omega(\xi)$ give a solution of Problem 1.

Corollary 1. *Let a solution of Problem 1 exist, and let the mapping function be rational:*

$$\omega(\xi) = c_0\xi + P_{n-l}(\xi)/Q_n(\xi), \tag{8}$$

where $P_{n-l}(\xi)$, $Q_n(\xi)$ are relatively prime polynomials in ξ of degrees $n-l$ and n, respectively.

Then the equality necessarily holds ($\tau \in C$)

$$F(\omega(\tau), \overline{\omega(\tau)})\omega'(\tau) = \Omega^-(\tau) + \overline{\omega(\tau)} N_s(\tau^{-1}), \tag{9}$$

where $\Omega^-(\tau)$ is the boundary value on C of a function analytic in K^- and equal to zero at infinity, and $N_s(\tau^{-1})$ is a polynomial in τ^{-1} of degree $s \le 2n$.

Proof. Substituting (8) into (7), taking into account the second of conditions (5), we obtain equality (9).

Remark 2. Condition (9) is necessary and sufficient for the normal solvability of equation (7) with respect to $\mu(\tau)$ [7].

Corollary 2. *Let $\omega(\tau)$ be a function analytic in $K^- + C$, and let representation (9) hold; then $\omega(\xi)$ is a rational function.*

The **proof** follows from [10], if (9) is substituted into (2).

Institute of Hydrodynamics
Siberian Branch of the Academy of Sciences of the USSR

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References

1. C. Miranda, *Partial Differential Equations of Elliptic Type*, IL, 1957.

2. L. A. Galin, PMM, **24**, issue 3 (1946).
3. G. N. Savin, *Stress Concentration around Holes*, 1951.
4. G. P. Cherepanov, PMM, **28**, issue 1 (1964).
5. R. Notrot, R. Timman, J. Eng. Math., **1**, No. 1 (1967).
6. V. V. Sokolovskii, *Theory of Plasticity*, 1950.
7. N. T. Khop, *Differential Equations*, **2**, No. 2 (1966).
8. A. V. Bitsadze, *Boundary Value Problems for Elliptic Equations of Second Order*, "Nauka," 1966.
9. F. D. Gakhov, *Boundary Value Problems*, Moscow, 1963.
10. N. E. Tovmasyan, *Differential Equations*, **2**, No. 2 (1966).

Note: Figure translations are in progress. See original paper for figures.