EXISTENCE AND ALMOST EVERYWHERE REGULARITY OF MINIMAL COMPACTS WITH PRESCRIBED HOMOLOGICAL PROPERTIES

MATHEMATICS

1969

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Abstract

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UDC 513.836

MATHEMATICS

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EXISTENCE AND ALMOST EVERYWHERE REGULARITY OF MINIMAL COMPACTS WITH PRESCRIBED HOMOLOGICAL PROPERTIES

(Presented by Academician P. S. Aleksandrov on 26 XII 1968)

1. An essential advance in the solution of the multidimensional Plateau problem was the theorem on the existence and almost everywhere regularity of a compact set X_0 realizing the absolute minimum of the Hausdorff measure $\Lambda^k(X \setminus A)$ over all compact sets X "spanning," in a certain precise sense, the compact set $A \subset E^n$, where E^n is Euclidean space. This theorem, proved by Reifenberg ($^{1-3}$), was generalized by Morrey to the case of Riemannian manifolds (^{4,5}). The possibility of these results lies in the new definition of boundary proposed by J. F. Adams and Reifenberg and carefully studied by Adams in the "Appendix" to (^{1}).

In the present note a theorem is formulated on the existence and almost everywhere regularity of a minimal compact set in the class of all compact sets realizing a subgroup $\mathcal{L}'$ in $H_k(\mathfrak{M}^n)$, where $\mathfrak{M}^n$ is a compact Riemannian manifold.

2. By $H_*(X, \mathfrak{G})$ we shall denote the Čech-Aleksandrov homology of the space X with coefficients in the Abelian group $\mathfrak{G}$.

Definition 1. Let $k > 1$, and let A be an arbitrary but fixed compact set in $\mathfrak{M}^n$; $X \subset \mathfrak{M}^n$. The **algebraic boundary** $b(X, A, \mathfrak{G})$ of the compact set $X \supset A$ with respect to A in dimension k is the group $\operatorname{Ker} i_*$, where

$$i_* : H_{k-1}(A, \mathfrak{G}) \to A_{k-1}(X, \mathfrak{G})$$

is the homomorphism induced by the inclusion $i : A \to X$.

Definition 2. Let $\mathcal{L}$ be a subgroup in $H_{k-1}(A)$; $\mathcal{L} \neq 0$; $A \subset \mathfrak{M}^n$. The **class** $\mathfrak{G}^k(A, \mathcal{L})$ is the totality of all compact sets $X \subset \mathfrak{M}^n$ such that $b(X, A, \mathfrak{G}) \supset \mathcal{L}$. (The case $\mathcal{L} = 0$ is uninteresting.)

Remark. Morrey' s existence theorem is the existence theorem in the class $\mathfrak{G}^k(A, \mathcal{L})$.

3. It turns out that in $\mathfrak{G}^k(A, \mathcal{L})$ it is reasonable to distinguish subclasses "parametrized" by subgroups $\mathcal{L}' \subset H_k(\mathfrak{M}^n, \mathfrak{G})$.

Definition 3. Let $\mathcal{L}' \subset H_k(\mathfrak{M}^n, \mathfrak{G})$; $A \subset \mathfrak{M}^n$. To the class

$$O^k(A, \mathcal{L}, \mathcal{L}')$$

we assign all compact sets $X \in \mathfrak{G}^k(A, \mathcal{L})$ such that $\alpha_* H_k(X) \supset \mathcal{L}'$, where α_* is the homomorphism induced by the inclusion $\alpha : X \to \mathfrak{M}^n$.

Ignoring the presence of $\mathcal{L}'$, i.e., formally putting $\mathcal{L}' = 0$, we obtain the equality $O^k(A, \mathcal{L}, 0) = \mathfrak{G}^k(A, \mathcal{L})$; ignoring the existence of $\mathcal{L}$, i.e., formally putting $\mathcal{L} = 0$ (which was meaningless in $\mathfrak{G}^k(A, \mathcal{L})$), we obtain classes $O^k(A, 0, \mathcal{L}')$, whose elements are connected with A only by the inclusion relation $X \supset A$; but if $A = \emptyset$, the classes $O_k(\emptyset, 0, \mathcal{L}')$ consist of compact sets "freely" realizing $\mathcal{L}'$.

4. **Main theorem.** *Let $\mathfrak{M}^n$ be a compact Riemannian manifold of class C^p, where $p \geq 4$; let A be an arbitrary compact set in $\mathfrak{M}^n$; let $\mathfrak{G}$ be either a compact Abelian group or a finite-dimensional vector space over some field F; let k be an integer and $k \geq 3$; and let $\mathcal{L}$ and $\mathcal{L}'$ be subgroups in $H_{k-1}(A, \mathfrak{G})$ and $H_k(\mathfrak{M}^n, \mathfrak{G})$, respectively, at least one of them being nontrivial. Suppose that $O^k(A, \mathcal{L}, \mathcal{L}') \neq \emptyset$ and that $d(A, \mathcal{L}, \mathcal{L}') < \infty$, where*

$$d(A, \mathcal{L}, \mathcal{L}') = \inf \Lambda^k(X \setminus A); \quad X \in O^k(A, \mathcal{L}, \mathcal{L}').$$

Then there exists a compact set $X_0 \in O^k(A, \mathcal{L}, \mathcal{L}')$ such that $\Lambda^k(X_0 \setminus A) = d(A, \mathcal{L}, \mathcal{L}')$, and every point $x \in (X_0 \setminus A) \setminus Z$, where $\Lambda^k(Z) = 0$, has in $X_0 \setminus A$ a neighborhood homeomorphic to a k-dimensional disk D^k. Moreover, if $\mathfrak{M}^n \in C^4$, then these k-disks may be assumed to belong to the class C^3_μ for any $0 < \mu < 1$; if $\mathfrak{M}^n \in C^p_\mu$ for some $p \geq 4$ and some $0 < \mu < 1$, then the k-disks may be assumed to belong to the class C^p_μ; finally, if $\mathfrak{M}^n \in C^\infty$ or is analytic, then the k-disks may be assumed to belong to the class C^∞ or to be analytic, respectively.

Putting $\mathcal{L}' = 0$, we obtain the theorem of Morrey; if, however, $A = \emptyset$, we obtain an existence theorem for "free cycles."

5. In the proof the following three theorems are used:

Theorem 1. Each class $O^k(A, \mathcal{L}, \mathcal{L}')$ is closed with respect to pointwise convergence $\lim_{n\to\infty} X_n$. By $\lim_{n\to\infty} X_n$ we mean the set of limit points of all possible sequences $\{x_n\}$, where $x_n \in X_n$, $n = 1, 2, \ldots$.

Theorem 2 (Adams). Let $X \in \mathfrak{G}^k(A, \mathcal{L})$, G be open in $\mathfrak{M}^n$; $A \cap \overline{G} = \emptyset$; $X \cap \partial G = B$; U be compact, $U \subset \overline{G}$; $U \cap \partial G = B$; $b(U, B) \supset b(X_1, B)$, where

$X_1 = X \cap \overline{G}$. Then the compact set X', obtained from X by replacing $X \cap G$ by $U \cap G$, again belongs to $\mathfrak{G}^k(A, \mathcal{L})$.

Theorem 3 (Adams). Let $A = A_1 \cup A_2$, where A_1 and A_2 are compact sets, $A_1 \cap A_2 = D$, $C_1 = A_1 \cup B$, $C_2 = A_2 \cup B$, $C = C_1 \cup C_2 = A \cup B$, where B is such a compact set that $b(B, D) \supset \overline{H}_{k-1}(D)$, $k \geq 3$. Then

$$i(C, C_1)_* H_k(C_1) + i(C, C_2)_* H_k(C_2) \supset i(C, A)_* H_k(A),$$

where $i(X, Y)$ is the embedding $Y \to X$.

The main observation consists in the fact that Theorem 3 can be applied to prove the closedness of the classes $O^k(A, \mathcal{L}, \mathcal{L}')$ with respect to deformations of a special kind, namely: cutting off long "whiskers" of the compact set X, which have little effect on the measure Λ^k, and replacing them by flat caps.

6. The construction of the minimal compact set X_0 is carried out as follows. Let $R(P) = \min(\rho(P, A), R_0)$, where P is a point in $\mathfrak{M}^n$; $\rho(x, y)$ is the distance in $\mathfrak{M}^n$; R_0 is some positive number independent of the point. Define

$$\varphi(r, P, X) = \int_0^r \Lambda^{k-1}[X \cap \partial B(P, t)]\, dt; \qquad 0 < R(P), \quad 0 < r < R(P);$$

$B(P, t)$ is the open ball of radius t with center at P; $\psi(r, P, X) = \Lambda^k[X \cap \overline{B}(P, r)]$.

Let $\{X_n\}$ be a minimizing sequence; $X_n \in O^k(A, \mathcal{L}, \mathcal{L}')$, $\Lambda^k(X_n \setminus A) = d(A, \mathcal{L}, \mathcal{L}') + \varepsilon_n$; $\{Q_i\}$ is a countable set dense in $\mathfrak{M}^n$. Then there exists a subsequence $\{X_{n'}\}$ such that the functions $\psi(r, Q_i, X_{n'})$ converge for every i and every r, $0 < r < R(P)$, to functions $\widetilde{\psi}(r, Q_i)$. Put

$$\psi^+(r, Q) = \lim_{\varepsilon \to 0} \sup_{\substack{r' < r + \varepsilon \\ \rho(Q, Q_i) < \varepsilon}} \widetilde{\psi}(r', Q_i); \qquad \psi^-(r, Q) = \lim_{\varepsilon \to 0} \inf_{\substack{r' > r - \varepsilon \\ \rho(Q, Q_i) < \varepsilon}} \widetilde{\psi}(r', Q_i);$$

$$\varphi^+(r, Q) = \limsup_{n \to \infty} \varphi(r, Q, X_n); \qquad \varphi^-(r, Q) = \liminf_{n \to \infty} \varphi(r, Q, X_n).$$

Let

$$\psi(P) = \lim_{r \to +0} [h(r)]^{-1} \psi^+(r, P), \qquad h(r) = \alpha(k) r^k (1 + hr)^{-k};$$

$\alpha(k)$ is the volume of the unit k-dimensional ball; h is some constant. Then, it turns out, $X_0 = S \cup A$, where $S = \{P \mid P \in \mathfrak{M}^n \setminus A,\ \psi(P) > 0\}$.

7. The proof of the main theorem is led by a chain of assertions, the most important of which are:

Theorem 4. Let $\mathfrak{M}^n$ be a compact Riemannian manifold of class C^p, $p \geq 4$, and $H_k(\mathfrak{M}^n) \neq 0$. Let $d' = \inf \Lambda^k(X)$, where $j_* H_k(X) \neq 0$; $j : X \to \mathfrak{M}^n$. Then $d' > 0$.

Theorem 5. The sequence $\{X_n\}$ can be modified in such a way that the new sequence $\{\widetilde{X}_n\}$ has the very same functions

$\psi^\pm$, $\widetilde{X}_n \in O^k(A, \mathcal{L}, \mathcal{L}')$, while the new functions $\varphi^\pm$ do not decrease in r, and $\bar{\rho}(\widetilde{X}_n, X_0) \to 0$ as $n \to \infty$, where

$$\bar{\rho}(\widetilde{X}_n, X_0) = \max_{x \in X_0} \rho(\widetilde{X}_n, x) + \max_{y \in \widetilde{X}_n} \rho(y, X_0).$$

The proof of local differentiability uses Theorem 3.

8. Corollary 1. Consider the class $R^k(A)$, where $X \in R^k(A)$ if $\alpha_* H_k(X) \neq 0$; $\alpha : X \to \mathfrak{M}^n$. Let $k \geq 3$, $\mathfrak{M}^n$ be a compact Riemannian manifold of class C^p, $p \geq 4$, $A \subset \mathfrak{M}^n$, $\mathfrak{G}$ be either a finit generated compact Abelian group or a finite-dimensional vector space over some field F. Then there exists a compact set $X_0' \in R^k(A)$ such that

$$\Lambda^k(X_0' \setminus A) = d'(A) = \inf_{X \in R^k(A)} \Lambda^k(X \setminus A)$$

and the local structure of this compact set is the same as for the compact sets of the main theorem. If $A \in R^k(A)$, then $d'(A) > 0$.

Consider the class $N^k(A)$, where $X \in N^k(A)$ if $b(X, A) \neq 0$. Then, if $H_{k-1}(A, \mathfrak{G})$ is either a finitely generated compact Abelian group or a finite-dimensional vector space over a field, there exists a minimal compact set $\widetilde{X}_0$ with analogous properties;

$$\tilde{d}(A) = \inf_{x \in N^k(A)} \Lambda^k(X \setminus A) = \Lambda^k(\widetilde{X}_0 \setminus A).$$

9. Along with the obvious relations

$$0 < d(A, \mathcal{L}, 0) \leq d(A, \mathcal{L}, \mathcal{L}') \geq d(A, 0, \mathcal{L}');$$

$$d(A, 0, \mathcal{L}') = 0, \quad \text{if } A \in O^k(A, 0, \mathcal{L}');$$

$$d(A, 0, \mathcal{L}') > 0, \quad \text{if } A \notin O^k(A, 0, \mathcal{L}'),$$

the following holds.

Corollary 2. Let $A = f(S^{k-1})$, where f is a homeomorphism; S^{k-1} is a sphere; $\mathfrak{G} = U$, the group of real numbers modulo 1; $\mathcal{L}$ and $\mathcal{L}'$ are nontrivial subgroups and $\mathcal{L}' \supset U$. Then either

$$\tilde{d}(A) < d(A, \mathcal{L}, \mathcal{L}'),$$

or

$$d'(A) < d(A, \mathcal{L}, \mathcal{L}').$$

With the compact sets $\widetilde{X}_0$ and X_0' one can associate a chain of groups $(\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4)$, where

$$\mathfrak{G}_1 = \operatorname{Im} H_k(\widetilde{X}_0), \quad \mathfrak{G}_2 = \operatorname{Im} H_k(X_0'), \quad \mathfrak{G}_3 = b(\widetilde{X}_0, A), \quad \mathfrak{G}_4 = b(X_0', A).$$

Corollary 3. For any compact Riemannian $\mathfrak{M}^n \in C^p$, $p \geq 4$, any homeomorphism f, and $\mathfrak{G} = U$, the chain $(\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4)$ is isomorphic to one of the following chains:

$$(\text{finite cyclic},\ \text{infinite cyclic},\ \neq 0,\ \neq \mathfrak{G}_4),$$

$$(\mathfrak{G}_1,\ U,\ \neq 0,\ 0),$$
$$(0,\ U,\ \neq 0,\ \neq 0),$$
$$(\supseteq U,\ \subseteq U,\ \neq 0,\ 0).$$

10. The notion of realization can also be defined by means of the homomorphism

$$\omega_* : \overline{H}_k(\mathfrak{M}^n, A) \to \overline{H}_k(\mathfrak{M}^n, S \cup A),$$

i.e., a compact set S realizes a subgroup $P \subset \overline{H}_k(\mathfrak{M}^n, A)$ if $\omega_* P = 0$. In some particular cases (for example, if $A = \emptyset$) the ω-variant of the minimum problem and the α-variant coincide. In (6) the ω-variant is investigated in the case when $\mathfrak{G}$ is a finitely generated Abelian group; A is a $(k-1)$-smoothable compact set; P is a subgroup generated by one element σ, and the minimum Λ^k is taken over k-smoothable compact sets.

I express my deep gratitude to Prof. P. K. Rashevskii, who drew my attention to this problem, and to Prof. E. G. Sklyarenko for valuable discussions.

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Received 20 XII 1968

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Note: Figure translations are in progress. See original paper for figures.

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