ON EIGENFUNCTIONS OF THE POLYHARMONIC OPERATOR

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Abstract

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MATHEMATICS

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ON EIGENFUNCTIONS OF THE POLYHARMONIC OPERATOR

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1°. Mean-value theorem. Let g be an arbitrary N-dimensional domain, $g \subset R_N$. Denote by $u(x)$ the classical solution in the domain g of the equation

$$\Delta^m u - (-1)^m \lambda u = 0; \tag{1}$$

m is any natural number; Δ^m is the polyharmonic operator; $\lambda \geqslant 0$. The eigenfunctions of the polyharmonic operator satisfy equation (1) for certain λ.

Theorem 1. Let $u(x)$ be any solution of equation (1) normalized in $L_2(g)$; let x be an arbitrary point of the open domain g; let r be any number smaller than the distance $\rho(x)$ from the point x to the boundary of the domain g; let r_0 be any number satisfying the condition $0 < r < r_0 < \rho(x)$; and let Γ_r be the sphere with center at the point x and radius r. Then the following equality holds:

$$\frac{1}{\omega_N}\int_{\Gamma_r} \Delta^{m-p} u\, d\omega = (-1)^{m-p} 2^{(N-2)/2}\Gamma\left(\frac{N}{2}\right)\mu^{2(m-p)}\frac{J_{(N-2)/2}(r\mu)}{(r\mu)^{(N-2)/2}}\, u(x) +$$

$$+ O\big(e^{-\beta_0(r_0-r)\mu}\big), \qquad p = 1, 2, \dots, m; \tag{2}$$

here $\mu = \lambda^{1/2m} > 0$; $\beta_0 = \min\{\beta_2, \dots, \beta_m\}$; $\beta_i = |\operatorname{Im}\sqrt{-\alpha_i}|$; α_i are the roots of degree m of $(-1)^m$; $\alpha_1 = -1$; $J_{(N-2)/2}$ is the Bessel function of order $(N-2)/2$, $\omega_N = \dfrac{2\pi^{N/2}}{\Gamma(N/2)}$ is the surface area of the unit sphere.

Formula (2) is a mean-value theorem over the sphere of radius r with center at the point $x \in g$ for solutions (when $p = m$) of equation (1) and for their Laplacians up to order $m - 1$. Theorem 1 is very essential in establishing the results presented below.

2°. Estimate of the maximum modulus of an eigenfunction.
Theorem 2. Let x be any fixed point of the open domain g. For the modulus of

any solution of equation (1) normalized in $L_2(g)$ (in particular, any eigenfunction of the polyharmonic operator), the estimate

$$|u(x)| = O\left(\lambda^{(N-1)/4m}\right), \qquad \lambda > 0 \text{ arbitrary.} \tag{3}$$

holds. This estimate is uniform in any strictly interior subdomain g_0 of the domain g.

Remark. As the example of the eigenfunctions of the N-dimensional ball shows, the estimate of the maximum modulus (3) is sharp.

3°. Summability by the Riesz means method of expansions in eigenfunctions. Let g be a bounded domain with sufficiently smooth boundary Γ. Denote by A any self-adjoint extension in $L_2(g)$ of the polyharmonic operator; $\{u_i(x)\}$ an orthonormal in $L_2(g)$ system of eigenfunctions of the operator A; $\{\lambda_i\}$ the corresponding system of eigenvalues; the sequence $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ can have only one limit point, $+\infty$.

We shall further denote by $\{E_\lambda\}$ the spectral family corresponding to the operator A, and by $e(\lambda, x, y)$ the kernel of the operator E_λ, called the spectral function of the operator A. The kernel of the operator

$$E_\lambda^\alpha = \int_0^\lambda \left(1 - \frac{\tau}{\lambda}\right)^\alpha dE_\tau$$

will be denoted by $e^\alpha(\lambda, x, y)$, $\alpha \geq 0$. The function $e^\alpha(\lambda, x, y)$, equal, if the operator A has a discrete spectrum (we consider here only this case), to

$$e^\alpha(\lambda, x, y) = \sum_{\lambda_i < \lambda} \left(1 - \frac{\lambda_i}{\lambda}\right)^\alpha u_i(x) u_i(y), \tag{4}$$

is called the Riesz kernel. It plays the same role in the summation of expansions in eigenfunctions by Riesz means as the spectral function $e(\lambda, x, y) = e^0(\lambda, x, y)$ does in the ordinary convergence of Fourier series;

$$e^\alpha(\lambda, x, f) = \int_g e^\alpha(\lambda, x, y) f(y)\, dy. \tag{5}$$

Theorem 3. 1) The Fourier series in eigenfunctions of the operator A of any function $f(x) \in L_p(g)$, $1 \leq p \leq 2$, is summed to this function at every Lebesgue point by Riesz means of order $\alpha > N/p - 1/2$, i.e. $e^\alpha(\lambda, x, f) \to f(x)$ as $\lambda \to \infty$ at every Lebesgue point, if $\alpha > N/p - 1/2$.

2) If $f(x) \in L_p(g)$, $1 \leq p \leq 2$, and has compact support in the domain g, then its Fourier series is summed to $f(x)$ at every Lebesgue point by Riesz means of order $\alpha > (N-1)/p$.*

Recall that a point $x \in g$ is called a Lebesgue point of a function $f(x) \in L_p(g)$ if

$$r^{-N} \int_{|x-y|\leq r} |f(x) - f(y)|^p \, dy = o(1)$$

as $r \to 0$. Almost all points of the domain g are Lebesgue points of the function $f(x)$.

4°. Asymptotics of the number of eigenvalues of the operator A. Denote by $N(\mu)$ the number of eigenvalues of the operator A not exceeding the constant μ:

$$N(\mu) = \sum_{\lambda_i < \mu} 1.$$

Theorem 4. For the number $N(\mu)$ of eigenvalues of the operator A not exceeding μ, the asymptotic formula

$$N(\mu) = \frac{\operatorname{mes} g}{2^N \pi^{N/2} \Gamma(N/2+1)} \, \mu^{N/2m} + O\Big(\mu^{(N-1)/2m} \ln^2 \mu\Big) \quad \text{as } \mu \to +\infty \quad (6)$$

is valid.

The asymptotic formula (6) is a strengthening, for the polyharmonic operator, of a recent work of Agmon ([3]), in which the remainder term in formula (6) for equations of order $2m$ with constant coefficients has the form $O(\mu^{(N-\sigma)/2m})$, where $\sigma < 1$. Theorem 4 is an extension to the case of the polyharmonic operator of a known result of Courant ([4]) for the eigenvalues of the Laplace operator, not improved until now for domains g of any rather general form. In general, the presence of the logarithm in the remainder term of an asymptotic formula of type (6) is, apparently, the limit to which one can advance in estimating $N(\mu)$ without a deep account of the properties of the boundary Γ and the character of the boundary conditions imposed in the eigenfunction problem. We note that for domains of general form Avakumović ([5]) succeeded in eliminating the logarithm in a formula of type (6) only in the case of the Laplace-Beltrami operator on a compact Riemannian manifold without boundary.

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* The last result, even for a general linear self-adjoint elliptic operator of order $2m$, was recently obtained by Hörmander; see ([1,2]).

Note: Figure translations are in progress. See original paper for figures.