MULTIVALUED MAPPINGS

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Abstract

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MATHEMATICS

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MULTIVALUED MAPPINGS

AND SPACES WITH A COUNTABLE NETWORK*

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Let X be a topological space. Consider a certain system $F_\pi(X)$ of closed subsets of the space X. The system $F_\pi(X)$ is complete if there exists a topological space $\widetilde{X} \supseteq X$, complete in the sense of Čech, for which $F_\pi(X) \subseteq F(\widetilde{X})$. By $F(\widetilde{X})$ we denote the collection of all closed subsets of the space $\widetilde{X}$.

Examples of complete systems:

1. The system $C(X)$ of nonempty bicompact subsets of the space X.
2. Let (X, ρ) be a metric space. Then the system

$$F_\pi(X) = \{L \in F(X) \mid L \text{ is complete in the metric } \rho\}$$

is complete.

In those cases when it is important to emphasize the complete system $F_\pi(X)$ of the space X, we shall use the notation $(X, F_\pi(X))$.

Let pairs $(X, F_\pi(X))$ and $(Y, F_\pi(Y))$ be given, where X and Y are completely normal spaces. The pairs $(X, F_\pi(X))$ and $(Y, F_\pi(Y))$ are (α, β)-homeomorphic if there exists a one-to-one mapping $f : X \to Y$ such that:

1) the mapping f is B-measurable** of class α;
2) the mapping f^{-1} is B-measurable of class β;
3) $F_\pi(Y) = \{fL \mid L \in F_\pi(X)\}$.

Let Z be some set and let $L_1(Z)$ be some algebra of subsets of the set Z (i.e., if $A, B \in L_1(Z)$, then $A \cup B$, $A \cap B$, $Z \setminus A \in L_1(Z)$). Put

$$S_1(Z) = \left\{ \bigcup_{n=1}^{\infty} A_n \mid A_n \in L_1(Z),\ n = 1, 2, ... \right\}.$$

Denote by $L_\beta(Z)$ the algebra of subsets generated by the system

$$\bigcup\{S_\alpha(Z) \mid \alpha < \beta\},$$

where $\beta < \omega_1$. Put

$$S_\beta(Z) = \left\{ \bigcup_{n=1}^{\infty} A_n \mid A_n \in L_\beta(Z),\ n = 1, 2, ... \right\}.$$

The pair $(X, F_\pi(X))$ satisfies property $l(\beta)$ (respectively, property $u(\beta)$) if, for every mapping $\theta : Z \to F_\pi(X)$, where $\theta^{-1}A \in S_1(Z)$ as soon as the set A is open (respectively, closed) in X, there exists a one-to-one mapping*** $\varphi : Z \to X$ such that:

1) $\varphi z \in \theta z$ for every point $z \in Z$;

2) for every open set U in X we have $\varphi^{-1}U \in S_\beta(Z)$.

In [7] the following is proved:

Theorem 1 (Kuratowski, Ryll-Nardzewski). *Let X be a separable metric space. Then, for any complete system $F_\pi(X)$ of the space X, the pair $(X, F_\pi(X))$ satisfies condition $l(1)$.*

The following lemmas are easily proved.

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* A system S of subsets of the space X is called a network in X if, for any x and Ox–a neighborhood of it in X–there is a $P \in S$ such that $x \in P \subseteq Ox$. The concept of a network was introduced by A. V. Arhangel' skii and successfully used by him in works [1-3]. Note that all spaces are assumed in advance to be completely regular.

** For the definition and basic properties of B-measurable mappings, see [6], p. 382.

*** Such mappings are called selections.

Lemma 1. Let the pair $(X, F_\pi(X))$, where X is a perfectly normal space, satisfy property $l(\beta)$. Then $(X, F_\pi(X))$ satisfies property $u(\beta)$.

Lemma 2. Let $(X, C(X))$, where X is a perfectly normal space, satisfy property $u(\beta)$. Then $(X, C(X))$ also satisfies property $l(1 + \beta)$.

Lemma 3. Let $g : X \to Y$ be a continuous perfect mapping. If the pair $(X, C(X))$ satisfies property $u(\beta)$, then the pair $(Y, C(Y))$ also satisfies property $u(\beta)$.

The following establishes a curious connection.

Theorem 2. Let the pairs $(X, F_\pi(X))$ and $(Y, F_\pi(Y))$ be $(0, \alpha)$-homeomorphic. If the pair $(Y, F_\pi(Y))$ satisfies property $l(\beta)$ (respectively property $u(\beta)$), then the pair $(X, F_\pi(X))$ satisfies property $l(\beta + \alpha)$ (respectively property $u(\beta + \alpha)$).

Proof. Let $f : X \to Y$ be a $(0, \alpha)$-homeomorphism for which

$$F_\pi(Y) = \{fL \mid L \in F_\pi(X)\}.$$

Let, further, $\theta : Z \to F_\pi(X)$ be such that $\theta^{-1}U \in S_1(Z)$ whenever U is open (respectively closed) in X. Consider $\psi : Z \to F_\pi(Y)$, where $\psi z = f(\theta z)$ for every point $z \in Z$. By the continuity of the mapping f, we have $\psi^{-1}V \in S_1(Z)$ whenever the set V is open (respectively closed) in Y. By hypothesis there exists a single-valued mapping $\varphi : Z \to Y$ such that $\varphi z \in \theta z$ and $\varphi^{-1}G \in S_\beta(Z)$ whenever the set G is open in Y. Put $g : Z \to X$, where

$$gz = f^{-1}(\varphi z).$$

Clearly $gz \in \theta z$ for every point $z \in Z$. Let U be an arbitrary open set of the space X. Since fU is a Borel set of class $\leq \alpha$, it follows that

$$\varphi^{-1}(fU) \in S_{\beta+\alpha}(Z).$$

In view of the equality

$$g^{-1}U = \varphi^{-1}(fU),$$

we have

$$g^{-1}U \in S_{\beta+\alpha}(Z).$$

This proves Theorem 2.

Theorem 3. Let X be a paracompact space with a σ-discrete net. For every complete system $F_\pi(X)$ of subsets of the space X there exists a continuous one-to-one mapping $f : X \to Y$, where (Y, ρ) is a metrizable space, such that: 1) fU is an F_σ-set for every open set U in X; 2) $f|L$ is a homeomorphism for every set $L \in F_\pi(X)$; 3) the set fL is complete with respect to the metric ρ for every set $L \in F_\pi(X)$. Moreover, if X has a countable net, then Y has a countable base.

Theorem 3 is easily derived from the following two propositions.

Proposition 1. Let $f : X \to Y$ be a one-to-one continuous mapping of a paracompact space X onto a metrizable space Y. For every complete system $F_\pi(X)$ there exists a one-to-one continuous mapping $g : X \to Z$ of the space X onto a metrizable space Z such that: 1) $g|L$ is a homeomorphism for every $L \in F_\pi(X)$; 2)

$$F_\pi(Z) = \{gA \mid A \in F_\pi(X)\}$$

is a complete system of closed subsets of the space Z.

Proof. Let $\widetilde{X}$ be a Čech-complete topological space for which $X \subseteq \widetilde{X}$ and $F_\pi(X) \subseteq F(X)$. Then there exists a paracompact Čech-complete space X' such that

$$X \subseteq X' \subseteq \widetilde{X}.$$

This fact is proved by the same methods as Theorem 5.8 in (3).

Clearly,

$$F_\pi(X) \subseteq F(X').$$

In this case there exists a perfect mapping $\varphi : X' \to S$, where S is a complete metrizable space (see (3, 8)). Consider $g : X \to Z \subseteq S \times Y$, where

$$gx = (\varphi x, fx)$$

for every point $x \in X$. By Lemma 1.5 of (5), $g|L$ is a homeomorphism for every set $L \in F_\pi(X)$. Let ρ be an arbitrary metric on Y and d some complete metric on S. In the space $S \times Y$ consider the metric

$$\mu((s,y),(s',y')) = d(s,s') + \rho(y,y').$$

It is easily verified that the set gL is complete with respect to the metric μ for every set $L \in F_\pi(X)$. Proposition 1 is proved.

Proposition 2. Let X be a perfectly normal paracompact space. Let, further,

$$\gamma = \{F_\alpha \mid \alpha \in A\}$$

be a discrete system

closed subsets of X. Then there exists a continuous mapping $f : X \to Y$, where Y is a metric space, such that the system $\omega = \{fF_\alpha \mid \alpha \in A\}$ is closed and discrete in X, and, moreover, the set F_α is marked* for every $\alpha \in A$.

Proof. By hypothesis there exists a discrete system $\Omega = \{U_\alpha \mid \alpha \in A\}$ of open subsets of the space X such that $F_\alpha \subseteq U_\alpha$ for every $\alpha \in A$. By the perfectly normality of the space X, for each $\alpha \in A$ there exists a continuous function $f_\alpha(x)$ such that

$$f_\alpha(x) = \begin{cases} 0, & \text{if } x \in X \setminus U_\alpha, \\ 1, & \text{if } x \in F_\alpha, \\ 0 < f_\alpha(x) < 1, & \text{if } x \in U_\alpha \setminus F_\alpha. \end{cases}$$

Consider the mapping $f : X \to S(A)^{**}$, where $fx = \{f_\alpha(x)\} \in S(A)$. Put

$$g_\alpha(\beta) = \begin{cases} 0, & \text{if } \beta \neq \alpha, \\ 1, & \text{if } \beta = \alpha. \end{cases}$$

It is obvious that $fF_\alpha = g_\alpha$ and $f^{-1}g_\alpha = F_\alpha$ for every $\alpha \in A$. Moreover, the system of points $\omega = \{g_\alpha = fF_\alpha \mid \alpha \in A\}$ is discrete. This proves Proposition 2.

Remark 1. In Proposition 2 it suffices to assume that X is collectively normal and that F_α is a G_δ-set for every $\alpha \in A$. Theorem 3, together with Theorems 1, 2 and Lemma 1, permits the following conclusion.

Theorem 4. *Let $F_\pi(X)$ be a complete system of subsets of a space X with a countable network. Then the pair $(X, F_\pi(X))$ satisfies the conditions $l(2), u(2)$.*

A single-valued mapping $\varphi : Y \to X$ is called an F_α-section (respectively, a G_α-section) for a multivalued mapping $\theta : Y \to F_\pi(X)$, if: 1) $\varphi y \in \theta y$ for every point $y \in Y$; 2) for every open set U in X, the set $\varphi^{-1}U$ belongs to the class*** $F_\alpha(Y)$ (respectively, to the class $G_\alpha(Y)$).

Theorem 5. *Let the pairs $(X, F_\pi(X))$ and $(Y, F_\pi(Y))$ be $(0, \alpha)$-homeomorphic. Further, let Z be some topological space. If for every continuous**** mapping $\theta : Z \to F_\pi(Y)$ there exists an F_β-section, then for every continuous mapping $\psi : Z \to F_\pi(X)$ there exists:*

a) *a $G_{\beta+\alpha-1}$-section, if β is even and α is odd and finite;*

b) *a $G_{\beta+\alpha}$-section, if β is odd and α is odd and finite;*

c) *a $G_{\beta+\alpha}$-section, if α is even and infinite;*

d) *an $F_{\beta+\alpha-1}$-section, if β is even and α is even and finite;*

e) *an $F_{\beta+\alpha}$-section, if β is odd and α is even and finite;*

f) *an $F_{\beta+\alpha}$-section, if α is odd and infinite.*

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* A set F is marked if $f^{-1}fF = F$. For properties of marked sets see [5].

** By $S(A)$ we denote the totality of all real functions defined on A with norm

$$\|g\| = \sum_{\beta \in A} |g(\beta)| < \infty$$

for every $g \in S(A)$.

*** Let Y be a topological space. $F_0(Y)$ is the family of closed G_δ-sets. The elements of the family $F_\alpha(Y)$ are intersections or unions of countable sequences of sets belonging to $\bigcup\{F_\beta(Y) \mid \beta < \alpha\}$, depending on whether the number α is even or odd. Put

$$G_\alpha(Y) = \{A = Y \setminus L \mid L \in F_\alpha(Y)\}.$$

**** A mapping $\theta : Z \to F_\pi(Y)$ is lower (upper) semicontinuous if, for every open (closed) set $A \subseteq Y$, the set $\theta^{-1}A = \{z \in Z \mid \theta z \cap A \neq \emptyset\}$ is open (closed) in the space Z. A mapping θ is continuous if it is simultaneously lower and upper semicontinuous (see [6,9]).

Proof. Let $f : X \to Y$ be a $(0, \alpha)$-homeomorphic mapping for which

$$F_\pi(Y) = \{fL \mid L \in F_\pi(X)\}.$$

Let, further, $\psi : Z \to F_\pi(X)$ be a continuous mapping. Consider $\theta : Z \to F_\pi(Y)$, where $\theta z = f(\psi z)$ for every point $z \in Z$. By the continuity of the mapping f, the mapping θ is continuous. Consequently, there exists an F_β-section $\varphi : Z \to Y$. Consider $g : Z \to X$, where $gz = f^{-1}(\varphi z)$ for every point $z \in Z$. It is obvious that g is a section for the mapping ψ. By elementary computations one can establish that the mapping g is the desired one.

The following theorem is proved analogously:

Theorem 6. Let the pairs $(X, F_\pi(X))$ and $(Y, F_\pi(Y))$ be $(0, \alpha)$-homeomorphic. Let, further, Z be some topological space. If for every lower semicontinuous (respectively, upper semicontinuous) mapping $\theta : Z \to F_\pi(Y)$ there exists an F_β-section, then also for every lower semicontinuous (respectively, upper semicontinuous) mapping $\psi : Z \to F_\pi(X)$ there exists a section satisfying conditions a)–f) of Theorem 5.

Remark 2. If, under the hypotheses of Theorem 5 or 6, for the mappings $\theta : Z \to F_\pi(Y)$ there exist G_β-sections, then for the mappings $\psi : Z \to F_\pi(X)$ there exist: a) $G_{\beta+\alpha}$-sections when α is even; b) $F_{\beta+\alpha}$-sections when α is odd.

In paper (9) a number of cases were established in which, for multivalued mappings in metric spaces, F_1-sections exist. Theorems 3, 5 and 6 and the corresponding theorems from (9) make it possible to construct G_2-sections for multivalued mappings in paracompact spaces with a σ-discrete network. For example, from the above-mentioned theorems and Theorem 2 from (9) there follows

Theorem 7. Let $F_\pi(X)$ be a complete system of a paracompact space X with a σ-discrete network. Then for every continuous mapping $\theta : Y \to F_\pi(X)$, where Y is a perfectly normal space, there exists a G_2-section.

In the same way, Theorems 6, 10, 14 and Corollary 2 from (9) carry over to paracompact spaces with a σ-discrete network. We note that Theorem 11 from (9) also carries over, but with greater difficulties.

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Note: Figure translations are in progress. See original paper for figures.