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Abstract

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ON THE SOLUTION OF DISCRETE WIENER-HOPF EQUATIONS IN A QUARTER-PLANE

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Discrete Wiener–Hopf equations in a quarter-plane have the form

$$\eta_{ij} = \sum_{k,l=0}^{\infty} a_{i-k,j-l}\xi_{kl}, \qquad ij = 0,1,2,\dots.$$

We shall assume that

$$\sum_{ij=0}^{\infty} |\eta_{ij}| < \infty, \qquad \sum_{p,q=-\infty}^{\infty} |a_{pq}| < \infty,$$

and seek a solution $\{\xi_{kl}\}_{k,l=0}$ also belonging to the space l_1 of sequences.

One-dimensional Wiener–Hopf equations on the half-line have been well studied (see ([1,2])). Multidimensional equations with a kernel depending on the difference of the arguments are solved in the whole space by applying the Fourier transform, and in the half-space they are essentially equations on a half-line with a parameter and are solved by the usual factorization method (see ([3,4])). Direct application of the factorization method to equations (1) is not possible, except in some very special cases, for example when the variables separate, i.e.

$$\sum_{p,q=-\infty}^{\infty} a_{pq}x^p y^q = a(x,y) = a(x)\tilde{a}(y),$$

in view of which a substantially new approach is required.

It turns out that there exists a procedure for explicitly solving these equations, at least for the case when there is an integer $N > 0$ such that $a_{pq} = 0$ if either $|p| > N$ or $|q| > N$. The idea of such an algorithm is presented in the present paper. We shall mainly restrict ourselves to the case $N = 1$.

Let us introduce some notation. Let $\mathfrak{R}$ be the ring of functions

$$r(x,y) = \sum_{i,j=-\infty}^{\infty} r_{ij} x^i y^j$$

on the torus $\{(x,y) : |x| = |y| = 1\}$ such that

$$\sum_{i,j=-\infty}^{\infty} |r_{ij}| < \infty;$$

$\mathfrak{R}_x^+$, $\mathfrak{R}_x^-$ are the subrings of functions $r(x,y)$ such that $r_{ij} = 0$, respectively, for $i < 0$ and $i \geq 0$; P_x^+, P_x^- are the projection operators onto these subrings. The R_y^+, R_y^-, etc. are defined analogously. Consider the operator

$$A\xi = \{\sum a_{i-k,j-l}\xi_{kl}\}_{i,j=0}$$

in the space l_1 of sequences $\{\xi_{kl}\}_{k,l=0}^{\infty}$, defined by the matrix $\|a_{pq}\|_{p,q=-\infty}^{\infty}$.

Theorem. The operator A is a Noether operator if and only if

$$a(x,y) \neq 0, \qquad |x| = |y| = 1; \tag{2}$$

$$\operatorname{ind}_{|x|=1} a(x,1) = \operatorname{ind}_{|y|=1} a(1,y) = 0. \tag{3}$$

Under these conditions the operator A is invertible, i.e., $\dim\ker A = \dim\operatorname{coker} A = 0$. The explicit inverse of the operator is given by the sequence of formulas (7), (8), (9), (14).

The first assertion of the theorem follows from the results of I. B. Simonenko on locally Noetherian operators (see ([7,8])) and the theory of multidimensional Wiener–Hopf equations in a half-space ([3,4]). The proof of the second assertion and the procedure for inverting the operator A are essentially given below. Put

$$\eta(x,y) = \sum_{ij=0}^{\infty} \eta_{ij} x^i y^j, \qquad \xi(x,y) = \sum_{k,l=0}^{\infty} \xi_{kl} x^k y^l;$$

$$\xi(y) = \xi(0,y), \qquad \xi(x,0) = \tilde{\xi}(x),$$

$$b(y) = a_{-1,1}y + a_{-1,0} + a_{-1,-1}\frac{1}{y}; \qquad \tilde{b}(x) = a_{1,-1}x + a_{0,-1} + a_{-1,-1}\frac{1}{x}.$$

Simple computations show that system (1) is equivalent to the following equation in generating functions (symbols):

$$\eta(x,y) = a(x,y)\xi(x,y) - \frac{1}{x}b(y)\xi(y) - \frac{1}{y}\tilde{b}(x)\tilde{\xi}(x) + a_{-1,-1}\frac{\xi_{00}}{xy}. \tag{4}$$

Lemma 1. System (1) has a solution if and only if there exist functions

$$\xi(y) = \sum_{j=0}^{\infty} \xi_j y^j; \qquad \tilde{\xi}(x) = \sum_{j=0}^{\infty} \tilde{\xi}_j x^j$$

and a constant ξ, with $\xi = \xi_0 = \tilde{\xi}_0$, such that

$$P_{\bar{y}} \left[\frac{1}{a_{\bar{y}}(x,y)} \left(\eta(x,y) + \frac{1}{x} b(y)\xi(y) + \frac{1}{y}\tilde{b}(x)\tilde{\xi}(x) - \frac{\xi a_{-1,1}}{xy} \right) \right] = 0, \tag{5}$$

$$P_{\bar{x}} \left[\frac{1}{a_{\bar{x}}(x,y)} \left(\eta(x,y) + \frac{1}{x} b(y)\xi(y) + \frac{1}{y}\tilde{b}(x)\tilde{\xi}(x) - \frac{\xi a_{-1,-1}}{xy} \right) \right] = 0. \tag{6}$$

Moreover, for any solution $(\tilde{\xi}(x), \xi(y), \xi)$ of system (5), (6), the solution of system (1) is found by the formula

$$\xi(x,y) = \frac{1}{a_y^+(x,y)} P_y^+ \left[\frac{1}{a_y^-(x,y)} \left(\eta(x,y) + \frac{1}{x} b(x)\xi(y) \right) \right]. \tag{7}$$

Here we have set

$$b(x,y) = \ln a(x,y), \qquad a_y^+(x,y) = a(x,y)/a_y^-(x,y),$$

$$a_y^-(x,y) = \exp\left[\sum_{\substack{j<0 \\ -\infty<i<\infty}} b_{ij} x^i y^j \right], \qquad b(x,y) = \sum_{i,j=-\infty}^{\infty} b_{ij} x^i y^j \in R.$$

The possibility of the latter representation under conditions (2) and (3) follows from the generalized Wiener theorem (see ([5])). We note that the application of the factorization method in Lemma 1 does not immediately lead to an explicit solution as in the one-dimensional case. The resulting system of two equations for functions of one variable explains the substantially greater complexity in comparison with the original Wiener–Hopf technique.

In our case*

$$a(x,y) = \frac{a_{11}x^2 + a_{01}x + a_{-1,1}}{xy}(y - y_0(x))(y - y_1(x)), \qquad a_y^-(x,y) = 1 - \frac{y_0(x)}{y},$$

where $y_0(x)$ and $y_1(x)$ are single-valued branches of an algebraic function, one of which, $y_0(x)$, takes values strictly inside the unit circle for $|x| = 1$, while the other, $y_1(x)$, lies strictly outside it. This follows easily from properties (2) and (3).

Substituting the expressions for $a_y^-(x,y)$ into (5), we obtain**

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* See the remark at the end of the paper.

** Using the following fact:

$$P_{\bar{y}}\left(\frac{1}{1-y_0/y}\,\omega(y)\right)=\omega(y_0)\sum_{k=1}^{\infty}\left(\frac{y_0}{y}\right)^k$$

for any $\omega(y)\in R_y^+$.

$$\sum_{k=1}^{\infty}y^{-k}\left(y_0^k\sum_{i=0}^{\infty}\omega_i y_0^i\frac{1}{x}+y_0^{k-1}\tilde{\omega}(x)+\frac{a_{-1,-1}\xi y_0^{k-1}}{x}+\eta(x,y_0)\right)=0, \qquad (8)$$

where

$$\omega(y)=\sum_{i=0}^{\infty}\omega_i y^i=b(y)\xi(y)-\frac{a_{-1,-1}\xi}{y}, \qquad \tilde{\omega}(x)=\tilde{b}(x)\tilde{\xi}(x)-\frac{a_{-1,-1}\xi}{x}.$$

Hence one can obtain the equivalent condition

$$\Omega(y_0(x))+\widetilde{\Omega}(x)=H(x), \qquad (9)$$

where we have denoted

$$\Omega(x)=x\omega(x), \qquad \widetilde{\Omega}(x)=x\tilde{\omega}(x)+a_{-1,-1}\xi, \qquad H(x)=-xy_0(x)\eta(x,y_0(x)).$$

Similarly, for equation (6) we obtain

$$\widetilde{\Omega}(x_0(y))+\Omega(y)=\widetilde{H}(y), \qquad (10)$$

where

$$\widetilde{H}(y)=-yx_0(y)\eta(x_0(y),y).$$

Lemma 2. The homogeneous system

$$\Omega(y_0(x))+\widetilde{\Omega}(x)=0, \qquad \widetilde{\Omega}(x_0(y))+\Omega(y)=0$$

has no nonconstant solutions $\Omega(x)$ and $\widetilde{\Omega}(x)$ belonging to the ring $\mathfrak{R}_x^+$.

Lemma 3. $\operatorname{ind} A=0$.

Proof. Under conditions (2) and (3), the function $a(x,y)$ realizes a mapping of the torus $\{(x,y):|x|=|y|=1\}$ homotopic to zero onto the plane with the point removed (the origin). This follows from the fact that the mapping of the torus to the unit circle $a(x,y)/|a(x,y)|$ corresponds to the zero element of the Bruschlinsky group (see ([12])). Thus the operator A can be connected by a path with the identity operator. Using now the fact that the index is a locally constant function on the set of Noetherian operators, we obtain the proof of the lemma.

Equation (9) resembles generalized Riemann boundary-value problems with shift for analytic functions, but it does not coincide with the cases that have been studied (see ([6,9])). Let Γ denote the unit circle in the plane of the complex variable C. One difficulty is that the image of Γ under the mapping $y_0(x)$ or $y_1(x)$ may have self-intersections. To avoid this, consider the following construction.

Let S_1 and S_2 be the Riemann surfaces of the functions $y(x)$ and $x(y)$, respectively, and let $h_1 : S_1 \to \overline{C}$ and $h_2 : S_2 \to \overline{C}$ be the natural branched coverings of the complex sphere $\overline{C}$; D is the interior of the unit disk. From the preceding it is not difficult to see that the covering paths $h_1^{-1}(\Gamma)$ and $h_2^{-1}(\Gamma)$ each consist of two nonintersecting simple closed contours on S_1 and S_2, respectively. $h_1^{-1}(\Gamma)$ is the boundary for the open set $h_1^{-1}(D) \subset S_1$, and $h_2^{-1}(\Gamma)$ for $h_2^{-1}(D) \subset S_2$.

It is known ([10]) that there exists a natural conformal one-to-one correspondence between the Riemann surfaces of the mutually inverse algebraic functions $y(x)$ and $x(y)$. We shall denote this correspondence by $f : S_1 \to S_2$.

Now we can transfer equations (9) and (10) to one of the surfaces, for example to S_2. Introduce the functions

$$\Omega_1(p) = \Omega(h_2(p)), \quad p \in h_2^{-1}(D);$$

$$\Omega_2(p) = \widetilde{\Omega}(h_1(f^{-1}(p))), \quad p \in fh_1^{-1}(D).$$

Note that one component of the boundary $fh_1^{-1}(\Gamma)$ of the set $fh_1^{-1}(D)$ will lie inside the set $h_2^{-1}(D)$, and the other outside it (we denote these components by Γ_1 and Γ_2, respectively). Similarly, one of the components of $h_2^{-1}(\Gamma)$ (denoted by $\tilde{\Gamma}_1$) lies inside $fh_1^{-1}(D)$, and the other $\tilde{\Gamma}_2$ outside it. This follows from the fact that, as was noted above, one of the branches $y(x)$ for $|x| = 1$ lies strictly inside the unit disk, and the other strictly outside it.

In the intersection of the domains (nonempty) $h_2^{-1}(D)$ and $fh_1^{-1}(D)$, equations (9) and (10) are equivalent and have the form

$$\Omega_1(p) + \Omega_2(p) = H^*(p), \tag{11}$$

where

$$H^*(p) = h^*(h_1 f^{-1}(p), h_2(p)); \qquad h^*(x,y) = -xyh(x,y).$$

Applying analyti-

tinuation, one can, with the aid of relation (11), extend the functions $\Omega_1(p)$ and $\Omega_2(p)$ to the domain $G = fh_1^{-1}(D) \cup h_2^{-1}(D)$, where they will also satisfy equation (11) (one may, without loss, assume that $h(x,y)$ is a polynomial and, consequently, is continued to G). The domain G is bounded by the curves Γ_2 and $\tilde{\Gamma}_2$.

To obtain an explicit integral representation of the solution, let us consider the representation of the torus as the factor group of its universal covering C by the period lattice $\{n\omega + n'\omega'\}$. For the given two-sheeted covering γ of the torus by the sphere S ($\gamma : S \to C$), there is defined an elliptic automorphism $\delta(\gamma)$ of the torus S, under which two points having the same image under the mapping γ are interchanged. From the properties of elliptic functions it is not hard to derive that, lifting $\delta(\gamma)$ to the universal covering C, we obtain (if the origin is shifted) an automorphism of C of the form $z \to -z$.

The preimages of the curves $\Gamma_{1,2}$ and $\tilde{\Gamma}_{1,2}$ on some fundamental parallelogram Π of the universal covering will be denoted by the same symbols with primes; a prime on a variable or a function will likewise denote the lift to the universal covering.

For the function $\Omega_1(p)$ the following relations hold; the first of them is determined by the lifting of $\Omega_1(p)$ to the Riemann surface, and the second follows from (11):

$$\begin{gathered} \Omega_1[(\delta(h_2))(p)] = \Omega_1(p); \\ \Omega_1[(\delta(h_1 f^{-1})(p)] - \Omega_1(p) = H^*[(\delta(h_1 f^{-1})(p)] - H^*(p). \end{gathered} \tag{12}$$

The product of the elliptic automorphisms $\delta(h_2)\delta(h_1 f^{-1})$ on C gives a shift by a certain vector $2b$ (where b is the vector between the centers of reflection for the two automorphisms). It can be shown that the curves $\Gamma'_{1,2}$, $\tilde{\Gamma}'_{1,2}$ are homologous and are elements of a canonical homology basis on the torus. Therefore

$$\begin{gathered} \Omega'_1(p') = \Omega'_1(p' + \omega), \quad p' \in C; \\ \Omega'_1(p' + 2b) - \Omega'_1(p') = H^*(p' + 2b) - H^*(p') \end{gathered} \tag{13}$$

(the second equation follows from both equations (12)).

As is not difficult to verify,* the solution of the system (13) is the function (14)

$$\Omega'_1(p') = \frac{1}{2\pi i} \int_{\Gamma'_2} \zeta(p' - t') \left[H^*(t' + 2b) - H^*(t')\right] dt', \tag{14}$$

where $\zeta(x)$ is the Weierstrass function corresponding to the periods ω and $2b$. This solution is unique (it must be holomorphic** in the domain bounded by the curves Γ_2' and $\Gamma_2' + 2b$) up to an additive constant, since the homogeneous system (13) defines a holomorphic elliptic function, which must be constant.

Remark. Above, we have essentially considered the case when the genus of the Riemann surface S_1, and hence also S_2, is equal to 1. The genus-0 case is simpler and is not considered here.

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* A similar problem for a rectangle was considered in ([11]).

** The pole can be removed by adding a suitable elliptic function.

Note: Figure translations are in progress. See original paper for figures.