

# ON THREE TYPES OF CONDITIONS FOR CONVERGENCE TO DIFFUSION PROCESSES

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## Abstract

## Full Text

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*MATHEMATICS*

Corresponding Member of the USSR Academy of Sciences A. A. BOROVKOV

# ON THREE TYPES OF CONDITIONS FOR CONVERGENCE TO DIFFUSION PROCESSES

In the present paper we consider general conditions for the convergence of arbitrary\* processes to Markov diffusion processes. The main object of the conditions is the increments of the process over relatively small time intervals. In essence, in problems on convergence to diffusion processes, in one form or another one has to require asymptotic regularity of the behavior of the first two moments of the increments. What is meant are moments conditional with respect to the  $\sigma$ -algebras generated by the histories. Thus, one is concerned with the proper asymptotic behavior of a sequence of random variables. Various approaches are possible here, but the essence of the conditions under consideration, and what they have in common, consists in requiring such regularity not on the entire space of elementary outcomes, but only on certain  $\omega$ -sets of sufficiently high probability (the requirement of asymptotic regularity of the moments of increments for every history is very stringent).

We shall distinguish here three ways of constructing convergence conditions.

- A. The introduction of special sets of states of the process, entry into which weakens the non-Markovian dependence of the moments on the history.
- B. The explicit selection of  $\omega$ -sets possessing the property indicated above.
- C. The introduction of conditions in which the required asymptotic regularity is required on the average.

Conditions of the first type are set forth in (<sup>4</sup>, <sup>5</sup>). Conditions of the second and third types are presented in this paper.

A. A characteristic feature of the processes considered in (<sup>4</sup>, <sup>5</sup>) is the presence of a set of states that is visited sufficiently often and such that the first two moments of the increments of the process over a large time interval after hitting this set would behave asymptotically like the moments of the corresponding diffusion process. The conditions ensuring compactness of the sequence of distributions under study are local in character and are connected with the cycles formed by the entries of the process into the indicated set. We note that such locality significantly simplifies the verification of convergence conditions. The

very assumption of the existence of cyclicity may seem somewhat artificial, although in essence it is only mildly restrictive. Nevertheless, the presence of a condition affecting the internal structure of the process is undesirable. Moreover, the very formulation of the convergence conditions in this case turns out to be quite complicated.

B. Conditions of the second type are simpler in form and contain no assumptions about the cyclicity of the process described above. The latter circumstance compels one to introduce a compactness condition (of the Lindeberg condition type) in a more difficult-to-check integral form.

Let  $\{Z(u), 0 \leq u \leq U\}$  be an arbitrary separable random process, defined in some space  $R(0, U)$  of real functions on  $[0, U]$  with a  $\sigma$ -algebra  $\mathfrak{B}_0^U$  containing the cylinder sets—

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\* For convergence of Markov processes, see the papers (1-3).

sets. If  $(\Omega, \mathfrak{M}, P)$  is the underlying probability space, then the process  $Z(u)$  is defined by specifying a measurable mapping of this space into  $(R(0, U), \mathfrak{B}_0^U)$ . We shall assume that the space of continuous functions  $C(0, U) \subset R(0, U)$ , and that the space of functions  $f_c(u) = f(cu)$ ,  $f \in R(0, U)$ , coincides with  $R(0, U)$ .

Next, let on  $(\Omega, \mathfrak{M})$  there be given a sequence of nested  $\sigma$ -algebras  $\mathfrak{M}(u)$ :  $\mathfrak{M}(u_1) \subset \mathfrak{M}(u_2)$  for  $u_1 \leq u_2$ , such that every event pertaining to the course of the process on the time interval  $[0, u]$  belongs to  $\mathfrak{M}(u)$ . In a number of cases it is convenient to regard  $\mathfrak{M}(u)$  simply as the  $\sigma$ -algebras generated by the trajectories  $Z(s)$  on  $[0, u]$ .

In what follows we shall consider a triangular-array scheme, i.e., a sequence of processes depending on a parameter  $T \rightarrow \infty$  and defined on increasing time intervals  $U = \nu T$ , where  $\nu > 0$  is fixed. We shall denote the process itself by  $Z_T(u)$ . Put

$$z_{u,t} = Z_T(u+t) - Z_T(u), \quad \xi(u) = Z_T(u)/\sqrt{T}.$$

By the letter  $t$  we shall denote sequences  $t = t(T) \rightarrow \infty$ ,  $t = o(T)$  as  $T \rightarrow \infty$ . The symbols  $N = N(T)$  and  $\varepsilon = \varepsilon(T)$  will denote sequences (not always the same ones) having the properties  $N \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$  as  $T \rightarrow \infty$ . Finally, let  $G_u \in \mathfrak{M}(u)$  denote the event

$$G_u = \{|\xi(u)| < N\}.$$

**Conditions BI-BIII.** Let there exist sequences  $\varepsilon, N$ , and  $t$ , and a system of sets  $\Omega_{u,t} \in \mathfrak{M}(u)$ ,  $0 \leq u \leq U$ , such that  $P\left(\bigcap_{k \leq U/t} \Omega_{kt,t}\right) \rightarrow 1$  as  $T \rightarrow \infty$ , and for every  $u \leq U - t$  on the set  $\Omega_{u,t} \cap G_u$ :

$$\text{BI.} \quad \mathbf{M}_{\mathfrak{M}(u)z_{u,t}} = \frac{t}{\sqrt{T}} [a(\xi(u)) + r_1(\omega)].$$

$$\text{BII.} \quad \mathbf{M}_{\mathfrak{M}(u)} z_{u,t}^2 = t [b(\xi(u)) + r_2(\omega)],$$

where  $a, b$  are functions whose properties will be specified later,

$$|r_1(\omega)| < \varepsilon, \quad |r_2(\omega)| < \varepsilon$$

almost everywhere in  $\omega \in \Omega$ .

$$\text{BIII.} \quad \mathbf{M}_{\mathfrak{M}(u)} (z_{u,t}^2; |z_{u,t}| > \delta\sqrt{T}) < \varepsilon t$$

for every  $\delta > 0$ .

It is natural also to assume that the relations BI–BIII, once satisfied, remain valid also for sequences  $t$  of the form  $t_1 = ct$ ,  $1/2 \leq c \leq 2$ .

Condition BIV concerns the functions  $a, b$ .

BIV. There exists a transition function  $P(x, u, B)$  of some Markov process such that the function

$$V(x, u) = \int \varphi(y) P(x, u, dy),$$

where  $\varphi$  is twice continuously differentiable, has continuous and bounded, in the region  $0 \leq u \leq \nu$ ,  $-\infty < x < \infty$ , derivatives  $\partial V/\partial u$ ,  $\partial V/\partial x$ ,  $\partial^2 V/\partial x^2$ , and is a solution of the Cauchy problem  $V(x, 0) = \varphi(x)$  for the equation

$$\frac{\partial V}{\partial u} = a \frac{\partial V}{\partial x} + \frac{b}{2} \frac{\partial^2 V}{\partial x^2}.$$

Moreover, there exists  $c > 0$  such that

$$|a(x)| \leq c(1 + |x|), \quad b(x) \leq c.$$

When this condition is fulfilled, there exists a diffusion process for which  $a$  and  $b$  are respectively the drift and diffusion coefficients.

A sufficient condition for BIV to hold may be the Hölder continuity of the functions  $a$  and  $b$  (6), or the boundedness and continuity of the derivatives  $a'$ ,  $b'$ ,  $a''$ ,  $b''$  (3).

Under the assumptions made, the following assertion is valid. Let

$$y_T(u) = Z_T(uT)/\sqrt{T}, \quad 0 \leq u \leq \nu,$$

and suppose there exists a limiting distribution  $y_T(0)$ , which we shall denote by  $p_0$ . Suppose, moreover, that  $\{w(u), 0 \leq u \leq \nu\}$  is a diffusion process in the space  $C(0, \nu)$  with the  $\sigma$ -algebra of Borel sets, constructed from the transition function  $P(x, u, B)$  and the initial distribution  $p_0$ .

**Theorem 1.** *If conditions BI–BIV are satisfied, the finite-dimensional distributions of  $y_T(u)$  converge to the finite-dimensional distributions of  $w(u)$ .*

The assertion of this theorem can be strengthened if one assumes that the following condition is fulfilled.

We shall call the value

$$J_{pZ}(u) = \lim_{\delta_1 \rightarrow 0, \delta_2 \rightarrow 0} \sup |Z_T(u + \delta_1) - Z_T(u - \delta_2)|.$$

the jump of the function  $Z_T(u)$  at the point  $u$ .

Denote by  $H_u^\delta$  the event

$$H_u^\delta = \{J_{pZ}(u) < \delta\sqrt{T}\}.$$

BV. The system of sets  $\Omega_{y,t}$  appearing in conditions BI–BIII, and the sets  $H_u^\delta$ , are such that\*

$$\mathbf{P} \left( \bigcap_{u \leq v} \Omega_{u,t} \right) \rightarrow 1, \quad \mathbf{P} \left( \bigcap_{u \leq v} H_u^\delta \right) \rightarrow 1$$

as  $T \rightarrow \infty$  for every  $\delta > 0$ .

Moreover, in those cases where for  $Z_T(u)$  there does not exist a stochastically equivalent process continuous from the right, the  $\sigma$ -algebras  $\mathfrak{M}(u)$  in conditions BI–BIII should be replaced by  $\mathfrak{M}(u+0)$ .

**Theorem 2.** *Let  $f$  be an  $(R, \mathfrak{B})$ -measurable functional on  $R(0, v)$ , continuous at the “points” of the space  $C(0, v)$  in the sense of the uniform metric:*

$$f(\psi_n) \rightarrow f(\psi), \quad \text{if } \rho_C(\psi_n, \psi) = \sup_{u \leq v} |\psi_n(u) - \psi(u)| \rightarrow 0, \\ \psi_n \in R(0, v), \quad \psi \in C(0, v).$$

*Then, if conditions BI–BV are satisfied, the distribution of  $f(y_T)$  converges to the distribution of  $f(w)$ .*

Let us note that if  $R(0, v)$  is a complete metric space with metric  $\rho_R$ , and if  $\rho_C(\psi_n, \psi) \rightarrow 0$  implies  $\rho_R(\psi_n, \psi) \rightarrow 0$ , then Theorem 2 implies convergence of the distributions of  $\rho_R$ -continuous functionals of  $y_T$  (weak convergence in  $R$ ).

C. Conditions of the third type have the following form.

Suppose there exists a sequence  $t \rightarrow \infty$ ,  $t = o(T)$ , such that for every  $\delta > 0$

$$\text{CI. } \mathbf{M} \left| \mathbf{M}_{\mathfrak{M}(u)z_{u,t}} - \frac{t}{\sqrt{T}} a(\zeta(u)) \right| < \varepsilon \frac{t}{\sqrt{T}}.$$

$$\text{CII. } \mathbf{M} \left| \mathbf{M}_{\mathfrak{M}(u)} z_{u,t}^2 - tb(\zeta(u)) \right| < \varepsilon t.$$

$$\text{CIII. } \mathbf{M} \left( z_{u,t}^2; |z_{u,t}| > \delta \sqrt{T} \right) < \varepsilon t.$$

Here  $\varepsilon \rightarrow 0$  as  $T \rightarrow \infty$ , and  $\delta > 0$  is arbitrary.

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\* Here it is assumed that the products of events under consideration are also events. The precise formulation of this condition requires the existence of measurable sets contained in these products and having probabilities converging to 1.

Conditions of a similar type (with averaging of deviations of conditional moments from their principal parts) were used by B. Rosen (7) in proving convergence to the normal law (see also (3)).

CIV. Coincides with BIV.

**Theorem 3.** *If the process  $Z_T(u)$  satisfies conditions CI–CIV, then the finite-dimensional distributions  $y_T(u)$  converge to the distributions  $w(u)$ .*

If, for the process  $Z_T(u)$ , the conditions of this theorem are fulfilled for  $\varepsilon = o(t/T)$ , then such a process will also satisfy the conditions of Theorem 1. The analogous converse assertion, evidently, does not hold.

In conclusion, we note that the proofs of Theorems 1, 3, instead of conditions I–III, essentially require the fulfillment of somewhat less restrictive relations. For example, in condition CII, instead of the relative closeness of  $\mathbf{M}_{\mathfrak{F}(u)} z_{u,t}^2$  and  $tb(\zeta(u))$ , uniformly for all intervals  $[u, u+t]$ , it is enough to require only that

$$\sum_{k < U/t} \mathbf{M} \left| \mathbf{M}_{\mathfrak{F}(kt)} z_{kt,t}^2 - tb(\zeta(kt)) \right| = o(T).$$

The conditions I, III also admit an analogous extension.

Institute of Mathematics  
Siberian Branch of the Academy of Sciences of the USSR  
Novosibirsk

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*Note: Figure translations are in progress. See original paper for figures.*

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