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PHYSICS

1969

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Abstract

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UDC 539.3:539.142

PHYSICS

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ON THE INERTIAL MOTION OF A BODY OF HOMOGENEOUS DEFORMATION

(Presented by Academician L. I. Sedov, 3 XII 1968)

§ 1. Let us consider the inertial motion of a body of homogeneous deformation (b.h.d.) as a mechanical system with 9 (apart from 3 translational) degrees of freedom $x_{ik}(t)$ (3 rotational and 6 deformational, $\varepsilon_{ik} = x_{ni}x_{nk} - \delta_{ik}$), according to the theory ⁽¹⁾, which is a generalization of the theory of an absolutely rigid body. In ⁽²⁾ it was shown that, for the special case of inertial motion—vortex-free motion ($2\Omega \equiv \text{rot } \mathbf{v} = 0$), with spherical symmetry of the undeformed body ($2j_k^0 = J_0$)—the formulas of the theory of b.h.d. coincide with the formulas of O. Bohr's theory ⁽³⁾ of collective motions in the atomic nucleus. There the case of pure rotation of the axes of deformation (rotation of the constant shape of the body under vortex-free motion of the substance) was also considered as an exact classical model corresponding to O. Bohr's theory, and a comparison was made with the motion of an ideal fluid in a rotating rigid shell—N. Zhukovsky's "elliptic rotation" ⁽⁴⁾, which O. Bohr calls "wave rotation." In the present work we set ourselves the task of considering the inertial motion of a b.h.d., abandoning the earlier ⁽²⁾ restrictions—the vortex-free character of the motion and the spherical symmetry of the body.

Introducing the matrix v_{ik} , inverse to the matrix x_{ik} , and the vectors

$$\mathbf{x}_k \equiv x_{ik}\mathbf{e}_i; \quad \mathbf{v}_k \equiv v_{ki}\mathbf{e}_i; \quad v_{ks}x_{ik} = \delta_{si}, \quad (1,1)$$

where \mathbf{e}_i are unit vectors along the axes of the inertial reference system $Ox_1x_2x_3$ with origin at the center of mass of the b.h.d., we write compactly the kinetic energy T and angular velocity Ω :

$$T = \frac{1}{2} \int v^2 dm = \frac{1}{2} j_k^0(\dot{\mathbf{x}}_k, \dot{\mathbf{x}}_k); \quad \Omega \equiv \frac{1}{2} \text{rot } \mathbf{v} = \frac{1}{2} [\mathbf{v}_k, \dot{\mathbf{x}}_k]. \quad (1,2)$$

The angular momentum of the b.h.d. is

$$\mathbf{M} = \int [\mathbf{r}, \mathbf{v}] dm = \mathbf{M}_{\text{rot}} + \mathbf{M}_{\text{df}}; \quad (1,3)$$

\mathbf{M}_{rot} is the angular momentum of rotation associated with Ω , namely

$$M_k^{\text{rot}} = I_{ki} \Omega_i; \quad I_{ki} = \int (r^2 \delta_{ki} - x_{kx} i) dm. \quad (1,4)$$

\mathbf{M}_{df} is the angular momentum of deformation, which is reduced to the form

$$\mathbf{M}_{\text{df}} = \frac{1}{2} j_k^0 \dot{\varepsilon}_{kn} [\mathbf{x}_k, \mathbf{v}_n], \quad (1,5)$$

i.e. $\mathbf{M}_{\text{df}} = 0$, if $\dot{\varepsilon}_{kn} = 0$. We take the internal energy U as the sum U' —the elastic energy of an isotropic body—and U_{surf} —the energy of the surface layer with S the coefficient of surface tension,

$$U = V_0 \left[\frac{1}{2} \lambda (\varepsilon^2 - \varepsilon_0^2) + \mu (\varepsilon_{ik} - \varepsilon_{ik}^0) (\varepsilon_{ik} - \varepsilon_{ik}^0) \right]; \quad V \equiv \frac{4}{3} \pi R_0^3. \quad (1,6)$$

ε_{ik}^0 is the static deformation produced by the surface tension

$$\varepsilon_{ik}^0 = \varepsilon_k^0 \delta_{ik}; \quad \varepsilon_k^0 \simeq 2 \frac{\mu_{\text{eff}}}{\mu'} \delta_k; \quad (1,7)$$

there is no summation over the underlined index. μ is the total shear modulus, equal to the sum of the elastic μ' and the effective $\mu_{\text{eff}} \equiv S(5R_0)^{-1}$. The surface of the s.u.d. in equilibrium is assumed to be an ellipsoid with semiaxes $R_0(1+\delta_k)$. R_0 is the radius of the equal-volume sphere; consequently, R_0 and two of the δ_k are independent. The limiting transition $\mu' \rightarrow 0$, signifying an ideal fluid, must be carried out simultaneously with $\delta_k \rightarrow 0$, since surface tension makes the equilibrium form spherical. The Lagrange function in the representation \varkappa_k has the form

$$L = \frac{1}{2} j_k^0 (\dot{\varkappa}_k, \dot{\varkappa}_k) - V_0 \left\{ \frac{1}{2} \lambda \varepsilon^2 + \mu [(\varkappa_i, \varkappa_k) - \delta_{ik}]^2 - 2\mu \varepsilon_k^0 (\varkappa_k, \varkappa_k) \right\}. \quad (1,8)$$

L has the form of the Lagrange function for three quasiparticles with effective masses j_k^0 in the dimensionless space \varkappa . The three degrees of freedom represented by the vector \varkappa are analogous to the three internal degrees of freedom of a diatomic molecule. The three \varkappa_k form a system of three interacting quasimolecules; the interaction energy of a pair is $W_{12} = V_0 \mu (\varkappa_1, \varkappa_2)^2$. We obtain the equations of motion in the following form

$$\ddot{\varkappa}_k + \frac{1}{2} \omega_{k0}^2 \left\{ [(\varkappa_k, \varkappa_n) - \delta_{kn}] \varkappa_n - \left(\varepsilon_k^0 - \frac{\lambda}{2\mu} \varepsilon \right) \varkappa_k \right\} = 0, \quad (1,9)$$

where the parameters ω_{k0} are determined by the equality

$$\omega_{k0}^2 \equiv 8V_0\mu(j_i^0)^{-1} \simeq \omega_0^2 \left(1 - \frac{2\mu}{\mu'}\delta_k\right); \quad \omega_0^2 \equiv 16V_0\mu J_0^{-1}. \quad (1.10)$$

§ 2. In the absence of deformation ($\varepsilon_{ik} = 0$), the \varkappa_k are three mutually perpendicular unit vectors; therefore, restricting ourselves to small deformations, one may seek the solution of the system (1.9) by successive approximations, using the substitution

$$\varkappa_k = \alpha_k + f_{ks}\alpha_s; \quad |f_{ks}| \ll 1 \quad (k = 1, 2, 3), \quad (2.1)$$

in which the α_k are three mutually perpendicular unit vectors, and, consequently, their change in time occurs according to the equations

$$\dot{\alpha}_k = [\omega, \alpha_k]. \quad (2.2)$$

Taking the three ω_k as the new unknown functions, we impose on the matrix f_{ks} the symmetry condition $f_{ks} = f_{sk}$, and then (2.1) will be a transformation from the nine functions $\varkappa_{ik}(t)$ to the nine $\omega_k(t), f_{ks}(t)$. For ε_{ik} we obtain

$$\varepsilon_{ik} = (\varkappa_i, \varkappa_k) - \delta_{ik} = 2f_{ik} + f_{in}f_{kn}, \quad (2.3)$$

and in the first approximation $\varepsilon_{ik} = 2f_{ik}$. Substituting (2.1) into (1.9), we obtain a system of 6 equations

$$\begin{aligned} \ddot{\varepsilon}_{ik} + (\omega_{ik}^2 - \omega^2)\varepsilon_{ik} - \omega_{ik}^2 \left(\varepsilon_i^0 - \frac{\lambda}{2\mu}\varepsilon \right) \delta_{ik} &= 2(\omega^2\delta_{ik} - \tilde{\omega}_i\tilde{\omega}_k) + \tilde{S}_{ik} + \tilde{S}_{ki} \\ &\quad - \{ \dot{\varepsilon}_{ks}(\omega, [\alpha_s, \alpha_i]) + \dot{\varepsilon}_{is}(\omega, [\alpha_s, \alpha_k]) \} \\ &\quad - \frac{1}{2}\tilde{\omega}_s(\varepsilon_{ks}\tilde{\omega}_i + \varepsilon_{is}\tilde{\omega}_k) \\ &\quad - \frac{1}{2}\{ \varepsilon_{ks}(\dot{\omega}, [\alpha_s, \alpha_i]) + \varepsilon_{is}(\dot{\omega}, [\alpha_s, \alpha_k]) \}; \\ 2\omega_{ik}^2 &\equiv \omega_{i0}^2 + \omega_{k0}^2 \end{aligned} \quad (2.4)$$

and a system of 3 equations, which are obtained by a cyclic permutation of the indices 1, 2, 3 from the equation

$$\begin{aligned} 2\dot{\tilde{\omega}}_1 + \frac{1}{2}(\omega_{20}^2 - \omega_{30}^2)\varepsilon_{23} &= -\varepsilon\tilde{\omega}_1 - \frac{1}{2}\dot{\varepsilon}\tilde{\omega}_1 + \dot{\varepsilon}_{1n}\tilde{\omega}_n + \frac{1}{2}\varepsilon_{1n}\dot{\tilde{\omega}}_n \\ &\quad - \frac{1}{2}\tilde{\omega}_n(\varepsilon_{2n}\tilde{\omega}_3 - \varepsilon_{3n}\tilde{\omega}_2) + \tilde{S}_{23} - \tilde{S}_{32}; \end{aligned} \quad (2.5)$$

$\tilde{\omega}_k, \dot{\tilde{\omega}}_k$ are the projections of $\omega, \dot{\omega}$ on the moving axes a_k . S_{ik} are expressions containing the small ε_{nm} in the second and higher powers. From (1,2) we obtain

$$\Omega = \omega - \frac{1}{2}L + \text{terms of higher order}; \quad L = L_k a_k; \quad (2,6)$$

$$L_1 = \frac{1}{4}(\varepsilon_{2n}\dot{\varepsilon}_{3n} - \dot{\varepsilon}_{2n}\varepsilon_{3n}); \quad L_2, L_3 \text{ by cyclic permutation.}$$

We solve the equations of motion (2,4), (2,5) by successive approximations, assuming: 1) second powers of ε_{ik} can be neglected in comparison with unity, 2) $\dot{\varepsilon}_{ik} \sim \omega_0 \varepsilon_{ik}$ and 3) $\omega, \dot{\omega} \ll \omega_0$.

In the first approximation the general solution of system (2,4) has the form

$$\varepsilon_{ik} = \varepsilon_{ik}^0 + A_{ik} \sin \omega_{ik} t + B_{ik} \cos \omega_{ik} t. \quad (2,7)$$

Substituting (2,7) into (2,5), one can find $\omega_k(t)$ and then, substituting them into (2,4), find ε_{ik} in the second approximation. We shall carry out this solution scheme in § 3 for a particular type of motion.

§ 3. Inertial motions of a body of homogeneous deformation for which the kinetic energy is conserved are of special interest, being an analogue of the motions of a rigid body. They may be called **generalized rigid motions**. Since in inertial motion the energy is conserved, for $T = \text{const}$ we shall also have $U = \text{const}$, and therefore the invariants of ε_{ik} must remain constant, i.e.

$$\varepsilon_k = \text{const} \quad (k = 1, 2, 3); \quad (3,1)$$

ε_k are the components of ε_{ik} reduced to diagonal form.

Generalized rigid motions are realized when the semiaxes of the ellipsoid of deformation are constant. They may be of the following three types: 1) proper rigid motions, when $\mathbf{M}_{\text{df}} = 0, \mathbf{M} = \mathbf{M}_{\text{rot}} = \text{const}$; 2) rotation of a constant form with irrotational motion of the matter (pure rotation of the axes of deformation), in which case $\mathbf{M}_{\text{rot}} = \Omega = 0, \mathbf{M} = \mathbf{M}_{\text{df}} = \text{const}$; 3) the general case, when $\mathbf{M}_{\text{df}} \neq 0, \mathbf{M}_{\text{rot}} \neq 0, \mathbf{M} = \mathbf{M}_{\text{rot}} + \mathbf{M}_{\text{df}} = \text{const}$.

Let us consider the possibility of realizing such motions. It is easy to see that already in the first approximation, i.e. for ε_{ik} of the form (2,7), such motions cannot exist if all ω_{k0} are different. Thus, if the equilibrium form of the body of homogeneous deformation is a triaxial ellipsoid, then generalized rigid motions do not exist. If, however, the equilibrium form of the body of homogeneous deformation is an ellipsoid of revolution: $\delta_1 = \delta_2; \delta_3 = -2\delta_1$ ($j_1^0 = j_2^0$), then one can satisfy condition (3,1); in this case we obtain

$$\varepsilon_{13} = \varepsilon_{23} = 0; \quad \varepsilon_{33} = -2\varepsilon_1^0; \quad (3,2)$$

$$\varepsilon_{11} = 2\varepsilon_1^0 - \varepsilon_{22} = \varepsilon_1^0 + A \sin \omega_{10}t + B \cos \omega_{10}t;$$

$$\varepsilon_{12} = \pm(B \sin \omega_{10}t - A \cos \omega_{10}t),$$

where A and B are arbitrary constants.

Is the character of the motion preserved in the next approximation? The second approximation must take into account in (2,4) the discarded terms containing $\tilde{\omega}_k$ and second powers of ε_{ik} . Taking account of the second powers of ε_{ik} makes the oscillations anharmonic, and T ceases to be constant. Let us consider the effect of $\tilde{\omega}_k$. Equations (2,5) for the case (3,2) become homogeneous and have the particular solution

$$\tilde{\omega}_1 = \tilde{\omega}_2 = 0; \quad \tilde{\omega}_3 = \omega = \text{const}; \quad \mathbf{a}_3 = \mathbf{e}_3. \quad (3,3)$$

For other solutions of system (2,5) there are no generalized rigid motions, whereas for (3,3) they can be preserved; substituting (3,3) into (2,4), we obtain we obtain

$$\varepsilon_{13} = \varepsilon_{23} = 0; \quad \varepsilon_{33} = -2C;$$

$$\varepsilon_{11} = 2C - \varepsilon_{22} = C + A \sin(\omega_{10} \pm \omega)t + B \cos(\omega_{10} \pm \omega)t; \quad (3,4)$$

$$\varepsilon_{12} = \mp[B \sin(\omega_{10} \pm \omega)t - A \cos(\omega_{10} \pm \omega)t]; \quad C = \varepsilon_1^0 + \frac{2}{3}(\omega/\omega_0)^2,$$

where A , B , ω are arbitrary constants. The tensor (3,4), reduced to diagonal form, has the form

$$\varepsilon_1 = C \pm \sqrt{A^2 + B^2}; \quad \varepsilon_2 = C \mp \sqrt{A^2 + B^2}; \quad \varepsilon_3 = -2C. \quad (3,5)$$

From (2,6) we find

$$\omega = \Omega \mp \frac{1}{4}e_3\omega_{10}(\varepsilon_1 - C)^2. \quad (3,6)$$

For the angular momenta we obtain the expressions:

$$\mathbf{M}_{\text{rot}} = e_3 I_{33} \Omega; \quad I_{33} = J_0 \left[1 + 2\delta_1 + \frac{2}{3} (\Omega/\omega_0)^2 \right]; \quad (3,7)$$

$$\mathbf{M}_{\text{df}} = \mp e_3 j_1^0 (\omega_{10} \pm \Omega) (\varepsilon_1 - C)^2; \quad 2j_1^0 = J_0 \left[1 + 2 \frac{\mu}{\mu'} \delta_1 + \left(\frac{\mu \delta_1}{\mu'} \right)^2 \right]. \quad (3,8)$$

The lower signs correspond to parallelism of \mathbf{M}_{rot} and \mathbf{M}_{df} , the upper signs to antiparallelism. The kinetic energy has the form

$$T = \frac{M_{\text{rot}}^2}{2I_{33}} + (\Omega, \mathbf{M}_{\text{df}}) + \frac{M_{\text{df}}^2}{2J_0(\varepsilon_1 - C)^2} \equiv T_{\text{rot}} + T_{\text{rd}} + T_{\text{df}}. \quad (3,9)$$

Introducing the total angular momentum M^2 , we give it the form

$$T = \frac{M^2}{2J_{\text{eff}}}; \quad J_{\text{eff}} \equiv I_{33} \left[1 + \frac{I_{33} M_{\text{df}}^2}{J_0 (\varepsilon_1 - C)^2 M^2} \right]^{-1}. \quad (3,10)$$

This formal notation conveniently embraces all three types of generalized rigid motions. Which of them are actually realized depends on the physical nature of the H.D. and on the ways in which it is acted upon. For example, an atomic nucleus may be excited by the Coulomb interaction with another nucleus flying past it; in this case only \mathbf{M}_{df} is excited ($\mathbf{M}_{\text{rot}} = 0$). This phenomenon is similar to a tidal wave (more precisely, to a “dent” wave, since Coulomb repulsion acts, not attraction). When excited by direct capture of a light nucleus by a heavy one, \mathbf{M}_{rot} is also excited.

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Received
22 VII 1968

REFERENCES CITED

1. V. G. Nevzglyadov, DAN, **141**, 1348 (1961); DAN, **142**, 59 (1962); *Nuovo Cim.*, **29**, 118 (1963).
2. V. G. Nevzglyadov, *Vestn. Leningrad. Univ.*, Ser. Math. and Mech., No. 19, 115 (1967).
3. A. Bohr, Collection *Problems of Contemporary Physics*, No. 1, 1956.
4. N. E. Zhukovskii, *Selected Works*, 1, 1948, pp. 31, 60.

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