ON ε-QUASICONFORMAL MAPPINGS OF A BALL ONTO A BALL

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Abstract

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MATHEMATICS

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ON ε-QUASICONFORMAL MAPPINGS OF A BALL ONTO A BALL

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It is known that for q-quasiconformal mappings in space with $q = 1 + \varepsilon$, smallness of ε guarantees the closeness of the mapping to a conformal one; and since the class of conformal mappings in space is exhausted only by Möbius transformations, under a corresponding normalization ε-quasiconformal mappings are close to the identity (see ([1–3])). However, all these results are mainly qualitative in character.

In the plane case, for ε-quasiconformal mappings of a disk onto a disk, P. P. Belinskii ([4]) obtained an exact estimate for the deviation of the mapping from a conformal one. He also ([5]) gave an estimate for the change of measure under mappings of this class.

In the present note analogous results are established for the case of ε-quasiconformal mappings of the three-dimensional ball onto itself. In addition, an estimate is given for the deviation of the mapping from a conformal one in the metric of the space W_2^1. However, the estimates obtained are not sharp.

1. Let the vector-function

$$y = f(x) \equiv (f_1(x), f_2(x), f_3(x)),$$

where $x = (x_1, x_2, x_3)$, realize an ε-quasiconformal mapping of the three-dimensional ball $B : |x| < 1$ onto itself, leaving fixed three fixed points on the boundary.

Theorem 1. *If the vector-function $y = f(x)$ realizes an ε-quasiconformal mapping of the three-dimensional ball B onto itself, normalized in the manner indicated above, then for small ε*

$$\|f(x) - x\|_{W_2^1} < \mathrm{const} \cdot \varepsilon^{1/4} + o(\varepsilon), \tag{1}$$

where the constant depends only on the normalization conditions, and by $\|f\|_{W_2^1}$ is denoted

$$\left(\sum_{i=1}^{3} \|f_i\|^2_{L^2(B)} + \sum_{i=1}^{3} \|\nabla f_i\|^2_{L^2(B)} \right)^{1/2}.$$

The proof of this theorem is carried out according to the following scheme. By virtue of a theorem of F. Gehring ([6]), the given mapping induces an ε-quasiconformal homeomorphism of the sphere $S : |x| = 1$ onto itself.

Hence, using the results of S. L. Krushkal' ([7]), it is not difficult to show that everywhere on the sphere S the inequality

$$|f(x) - x| < \text{const} \cdot \varepsilon + o(\varepsilon), \tag{2}$$

holds, where the constant depends only on the normalization conditions.

Consider $u_i(x)$ $(i = 1, 2, 3)$—harmonic functions coinciding on the boundary with the components of the vector-function $f(x)$, i.e. $u_i|_S = f_i|_S$ $(i = 1, 2, 3)$. It can be shown that for ε-quasiconformal mappings

$$\|\nabla(f - x)\|_{L^2} < \text{const} \cdot (\varepsilon + \|\nabla(u - x)\|_{L^2})^{1/2} + o(\varepsilon). \tag{3}$$

To estimate $\|\nabla(u - x)\|_{L^2}$ we use Green' s formula

$$\|\nabla(u - x)\|^2_{L^2} \equiv \sum_{i=1}^{3} \int_B |\nabla(u_i - x_i)|^2 \, dB = \sum_{i=1}^{3} \int_S (u_i - x_i) \, \frac{\partial(u_i - x_i)}{\partial n} \, dS,$$

where $\partial/\partial n$ denotes differentiation along the normal. Hence, by virtue of inequality (2), we obtain

$$\|\nabla(u - x)\|^2_{L^2} < \text{const} \cdot \varepsilon \sum_{i=1}^{3} \int_S \left| \frac{\partial(u_i - x_i)}{\partial n} \right| dS. \tag{4}$$

The uniform boundedness of the contour integrals in the right-hand side of (4) is established on the basis of M. I. Vishik' s inequality [8] for the normal derivatives of harmonic functions in a ball and the fact that f is quasiconformal on the sphere S.

Thus, from (3) and (4) we have

$$\|\nabla(f - x)\|_{L^2} < \text{const} \cdot \varepsilon^{1/4} + o(\varepsilon). \tag{5}$$

Inequality (5), together with relation (2), gives an analogous estimate for $\|f - x\|_{L^2}$, whence the validity of the theorem follows.

2. From Theorem 1, using the uniform continuity of ε-quasiconformal mappings of the ball onto itself (see, for example, [9]), we obtain:

Theorem 2. *For ε-quasiconformal mappings of the three-dimensional ball onto itself, normalized in the same way as above, the inequality*

$$\max_B |f(x) - x| < \text{const} \cdot \varepsilon^{1/16} + o(\varepsilon), \quad (6)$$

holds, where const *depends only on the normalization conditions.*

The following theorem gives a quantitative estimate of the change of measure under ε-quasiconformal mappings.

Theorem 3. *Let the vector function $y = f(x)$ realize an ε-quasiconformal homeomorphism of the three-dimensional ball B onto itself, normalized in the manner indicated above. Then, for every measurable $E \subset B$, the inequality*

$$|\operatorname{mes} f(E) - \operatorname{mes} E| < \text{const} \cdot \varepsilon^{1/2} + o(\varepsilon), \quad (7)$$

is valid, where the constant depends only on the normalization conditions.

The proof of this theorem is carried out essentially analogously to [5], using relation (2).

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Note: Figure translations are in progress. See original paper for figures.