ON THE THEORY OF AN OPERATOR CALCULUS

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Abstract

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MATHEMATICS

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ON THE THEORY OF AN OPERATOR CALCULUS

FOR LINEAR UNBOUNDED OPERATORS

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Let a be a closed linear operator with dense domain of definition $D(a)$ in a countably normed locally convex space E. Under certain restrictions the following formula is valid (λ is a complex number):

$$(\lambda - a)^{-1}x = \frac{1}{\lambda}x + \frac{1}{\lambda^2}ax + ... + \frac{1}{\lambda^n}a^{n-1}x + \frac{1}{\lambda^n}(\lambda - a)^{-1}a^n x. \qquad (1)$$

If in some sector Σ of the complex plane the behavior of the function $(\lambda - a)^{-1}x$ as $|\lambda| \to \infty$ is characterized by the first n terms of the expansion (1), then for the operator a one can construct an operator calculus generalizing the calculus of N. Dunford ([1]), Ch. VII, for bounded operators. In this case functions of a will be defined, generally speaking, only on the domain of definition of the operator a^{n-1}. The calculus constructed includes fractional powers of the operator a and semigroups generated by fractional powers.

For the case $n = 1$, an exposition of the theory of fractional powers of closed operators and analytic semigroups of linear continuous operators can be found in ([2–4]).

1. Restrictions on the operator. Let the numbers $\sigma > 0$ and $0 < \vartheta < \pi$ be fixed. Denote by Γ the contour in the complex plane composed of the two rays $\arg(\lambda - \sigma) = \pm\vartheta$, by Σ the closed sector containing the negative real semiaxis and bounded by the contour Γ, and by Σ' the closed sector whose interior is complementary to the sector Σ. Thus the sectors Σ and Σ' intersect along the contour Γ.

Suppose that the following assumptions are satisfied.

1°. For every $\lambda \in \Sigma$, the operator $\lambda - a$ establishes a one-to-one correspondence between the domain of definition $D(a)$ and its range $R(\lambda - a)$.

2°. For some natural number $n \geq 1$, the domain of definition $D(a^n)$ of the operator a^n is dense in the space E. Here, by definition, $x \in D(a^k)$ if $x \in D(a^{k-1})$ and $a^{k-1}x \in D(a)$, $k > 1$.

3°. For every $\lambda \in \Sigma$, the operator $(\lambda - a)^{-1}$ is weaker than the operator a^{n-1}, which means the following: $D((\lambda - a)^{-1}) \supset D(a^{n-1})$.

4°. The following inequality holds for all $\lambda \in \Sigma$ and $x \in D(a^{n-1})$:

$$|(\lambda - a)^{-1}x|_p \leq (1 + |\lambda|)^{-1}|x|_q + (1 + |\lambda|)^{-2}|ax|_q + \ldots$$
$$\ldots + (1 + |\lambda|)^{-n}|a^{n-1}x|_q. \quad (2)$$

Here and in what follows, expressions of the form $|\varphi x|_p \leq |\psi x|_q,\ x \in M$, are to be understood as follows: for every seminorm $p(x)$ continuous on E there exists a seminorm $q(x)$, continuous on E, such that for all elements x belonging to the set M, the inequality $p(\varphi x) \leq q(\psi x)$ holds, where φ and ψ are given operators in the space E.

2. Properties of the resolvent. It follows from 1°-2° that in the sector Σ the operator a cannot have eigenvalues or points of the residual spectrum. However, the spectrum of the operator a may fill the whole plane.

Let the element $x \in D(a^n)$. Then the identity holds

$$x = \lambda^{-1}(\lambda - a)x + \lambda^{-2}(\lambda - a)ax + \cdots$$
$$\cdots + \lambda^{-n}(\lambda - a)a^{n-1}x + \lambda^{-n}a^n x \quad (\lambda \in \Sigma). \quad (3)$$

Since $D(a^n) \subset D(a^{n-1})$, by 3° the operator $(\lambda - a)^{-1}$ can be applied to the element x. To all terms on the right, except the last, this operator can obviously also be applied. Consequently, the operator $(\lambda - a)^{-1}$ can be applied to all terms of this identity, which gives the expansion (1). Further, in the relation $x - (\mu - a)(\lambda - a)^{-1}x = (\lambda - \mu)(\lambda - a)^{-1}x$, $x \in D(a^{n-1})$, the operator $(\mu - a)^{-1}$ can be applied to the difference on the left, and hence it can also be applied to the right-hand side. As a result one obtains the **Hilbert identity**

$$(\mu - a)^{-1}x - (\lambda - a)^{-1}x = (\lambda - \mu)(\mu - a)^{-1}(\lambda - a)^{-1}x$$

$$(\lambda, \mu \in \Sigma,\ x \in D(a^{n-1})). \quad (4)$$

Writing it for $\mu = 0$, by successive application of the operator a we see that $(\lambda - a)^{-1}x \in D(a^{n-1})$, if $x \in D(a^{n-1})$, i.e., the operator $(\lambda - a)^{-1}$ maps the set $D(a^{n-1})$ into itself. Thus, for every $x \in D(a^{n-1})$ the square of the resolvent is defined—the element $(\lambda - a)^{-2}x$. From the Hilbert identity there now follows the existence of the derivative of the function $(\lambda - a)^{-1}x$ (equal to $-(\lambda - a)^{-2}x$) and, therefore, its analyticity in the sector Σ.

3. **The main theorem of the operational calculus.** Denote by $\mathcal{F}$ the family of numerical functions $\varphi(\lambda)$ of the complex variable λ, analytic in the sector Σ' and satisfying the condition $|\varphi(\lambda)| \le c|\lambda|^{-\gamma}$, $\lambda \in \Sigma'$, where the positive constants c and γ, generally speaking, depend on the function φ.

Definition. Let $\varphi(\lambda) \in \mathcal{F}$. Then the operator $\varphi(a)$ on elements of the set $D(a^{n-1})$ is defined by the equality

$$\varphi(a)x = \frac{1}{2\pi i}\int_\Gamma \varphi(\lambda)(\lambda - a)^{-1}x\,d\lambda \quad (x \in D(a^{n-1})), \tag{5}$$

where the contour Γ is traversed so that the sector Σ' remains on the left.

From the assumptions made, the absolute convergence of this integral follows. The resulting operators need not be closed. In the general case, however, the set $D(a^{n-1})$ turns out to be a natural domain of definition of the operators $\varphi(a)$.

Theorem 1. Let $\varphi, \psi \in \mathcal{F}$ and let α, β be complex numbers. Then:

(a) $\alpha\varphi + \beta\psi \in \mathcal{F}$ and

$$(\alpha\varphi + \beta\psi)(a)x = \alpha\varphi(a)x + \beta\psi(a)x, \quad x \in D(a^{n-1});$$

(b) $\varphi\psi \in \mathcal{F}$ and

$$(\varphi\psi)(a)x = \varphi(a)\psi(a)x,$$

if $\psi(a)x \in D(a^{n-1})$;

(c) let a family of functions $\varphi_z(\lambda) \in \mathcal{F}$, depending on the parameter z, be uniformly bounded:

$$|\varphi_z(\lambda)| \le c,$$

and converge to the function $\varphi_0(\lambda) \equiv 1$ as $z \to 0$, uniformly in λ on each compact subset of the sector Σ'. Then for any fixed $x \in D(a^n)$

$$\lim \varphi_z(a)x = x \quad \text{as } z \to 0. \tag{6}$$

If, in addition, the functions $\varphi_z(a)x$ are uniformly bounded,

$$|\varphi_z(a)x|_p \le |x|_r + |z|(|ax|_r + \cdots + |a^{n-1}x|_r), \quad x \in D(a^{n-1}), \tag{7}$$

then the limiting relation (6) is valid for all $x \in D(a^{n-1})$.

Proof. Part (a) is obvious; (b) is proved in the usual way through the Hilbert identity (4). We dwell on (c). If

$x \in D(a^n)$, then $ax \in D(a^{n-1})$, and one can write

$$x - \varphi_z(a)x = (a^{-1} - \varphi_z(a)a^{-1})ax = \frac{1}{2\pi i}\int_\Gamma \lambda^{-1}(1 - \varphi_z(\lambda))(\lambda - a)^{-1}ax\, d\lambda.$$

By virtue of the assumptions made, the integral here converges absolutely and uniformly in z; therefore, for a fixed seminorm $p(x)$, it can be replaced, to within an arbitrary $\varepsilon > 0$, by an integral over a bounded contour, and this latter can be made less than ε by choosing z sufficiently small. This gives relation (6).

If inequality (7) is satisfied, then for any $x, y \in D(a^{n-1})$

$$|x - \varphi_z(a)x|_p \leq |x - y|_p + |y - \varphi_z(a)y|_p + |x - y|_r + |z| \sum |a^k(x - y)|_r.$$

Now let $y \in D(a^n)$. Since this set is dense in E, for fixed seminorms $p(x)$ and $r(x)$ one can find such a y that $|x - y|_p < \varepsilon$ and $|x - y|_r < \varepsilon$. We seek $\delta > 0$ such that for all $|z| < \delta$ the relations $|y - \varphi_z(a)y|_p < \varepsilon$ and $|z| \sum |a^k(x-y)|_r < \varepsilon$ hold. The first can be achieved by virtue of (6), and in the second the sum is fixed. This proves the last assertion of the theorem.

4. **Analytic semigroups.** If the angle $\vartheta < \pi/2$, i.e., the sector Σ' lies in the right half-plane, then for the operator $-a$ the exponential function $u(z)x$, defined by formula (5), is defined, where $\varphi(\lambda) = \exp(-z\lambda)$. Here the argument z ranges over the open sector Δ: $|\arg z| < \pi/2 - \vartheta$. The function introduced is not, in general, a continuous operator for every $z \in \Delta$. Nevertheless it possesses the basic properties of analytic semigroups of linear operators. Namely, directly from the definition it is clear that for $x \in D(a^{n-1})$ the function $u(z)x$ is analytic in z in the sector Δ, and

$$\frac{d}{dz}u(z)x = -au(z)x, \qquad u(z_1 + z_2)x = u(z_1)u(z_2)x \quad (x \in D(a^{n-1})).$$

The latter follows from part (b) of Theorem 1.

Finally, if in relation (8), taking $z \in \Delta$ sufficiently small, the contour Γ is shifted to the left so that for $\lambda \in \Gamma$ the inequality $|\lambda|^{-1} \leq |z|$ holds, then estimate (7) is obtained. From part (c) of Theorem 1 it follows that the semigroup is strongly continuous at zero on the elements $x \in D(a^{n-1})$, if $|\arg z| \leq \vartheta - \varepsilon$ for any $\varepsilon > 0$. We have arrived at the following assertion:

Theorem 2. If for the closed operator a conditions 1°–4° are satisfied (with angle $\vartheta < \pi/2$), then the operator $-a$ generates a semigroup $u(z)x$, defined on the set of elements $x \in D(a^{n-1})$ dense in E, analytic in z in the open sector Δ and strongly continuous at zero.

5. **Fractional powers** of the operator a, satisfying conditions 1°–4°, are defined by the integral (5) with the function $\varphi(\lambda) = \lambda^{-z}$. It is assumed

here that $\operatorname{Re} z > 0$. If $z = k$, where k is an integer, then in this integral the contour of integration can be contracted to zero. By the residue theorem one obtains the coincidence of the integral written above with the element $a^{-k}x$. Assuming $|z| < 1$, we contract the contour of integration to the negative real axis. Then estimate (7) is obtained. According to part (c) of Theorem 1 this means that $\lim a^{-z}x = x$ as $z \to 0$ for $x \in D(a^{n-1})$. Thus we obtain

Theorem 3. Let the operator a satisfy conditions 1°–4°. Then the function $a^{-z}x$, defined by formula (5) with $\varphi(\lambda) = \lambda^{-z}$, is analytic in z in the open right half-plane and strongly continuous at zero on $D(a^{n-1})$. It forms a semigroup on the set $D(a^{2n-2})$, dense in the space E.

For values $0 < \alpha < \pi/2\vartheta$, in the open sector Δ_α: $|\arg z| < \pi/2 - \alpha\vartheta$, the function $\varphi(\lambda) = \exp(-z\lambda^\alpha)$ is defined and exponentially decreasing. The operator-valued function $u_\alpha(z)x$ constructed from it is analytic in z and satisfies the relations

$$\frac{d}{dz}u_\alpha(z)x = -a^\alpha u_\alpha(z)x, \qquad u_\alpha(z_1 + z_2)x = u_\alpha(z_1)u_\alpha(z_2)x.$$

Moreover, by Theorem 1(c),

$$\lim_{z\to 0} u_\alpha(z)x = x \quad \text{for } x \in D(a^n).$$

Since $u_\alpha(z)x \in D(a^k)$ for every $k > 0$, and the set $D(a^n)$ is dense in E, the last relation means that the sets $D(a^k)$ are also dense in E for every $k > 0$. We note that estimate (7), as in the case $n = 1$, can be proved only for exponents $0 < \alpha \le 1/2$.

6. Examples. Let

$$E = L_2(-\infty, \infty) \oplus L_2(-\infty, \infty),$$

with elements that are pairs of functions $x = (x_1(p), x_2(p))$. The operator a and its resolvent are given by the matrices

$$a = \begin{pmatrix} 0 & -p \\ p^{-1} & 0 \end{pmatrix}, \qquad (\lambda - a)^{-1} = \begin{pmatrix} \lambda(\lambda^2+1)^{-1} & -p(\lambda^2+1)^{-1} \\ p^{-1}(\lambda^2+1)^{-1} & \lambda(\lambda^2+1)^{-1} \end{pmatrix}.$$

The spectrum of a consists of two eigenvalues $\pm i$ of infinite multiplicity and a continuous spectrum filling the whole plane. In the present case

$$(\lambda - a)^{-1} = \lambda(\lambda^2+1)^{-1} + (\lambda^2+1)^{-1}a,$$

therefore, for any function $\varphi(\lambda) \in \mathcal{F}$,

$$\varphi(a)x = \frac{1}{2}[\varphi(i) + \varphi(-i)]x + \frac{1}{2}[\varphi(i) - \varphi(-i)]ax, \qquad x \in D(a).$$

Hence it is clear that, if the second term does not vanish, then the operator $\varphi(a)$ is defined on $D(a)$, and only there.

Let also the operator a and its resolvent be represented by the matrices

$$a = \begin{pmatrix} p^2 & 0 \\ -p^k & p^2 \end{pmatrix}, \qquad (\lambda - a)^{-1} = \begin{pmatrix} (\lambda - p^2)^{-1} & 0 \\ p^k(\lambda - p^2)^{-2} & (\lambda - p^2)^{-1} \end{pmatrix}.$$

It is evident that for $k > 4$ the resolvent is unbounded. From the identity

$$(\lambda - a)^{-1} = \begin{pmatrix} \lambda(\lambda - p^2)^{-2} & 0 \\ 0 & \lambda(\lambda - p^2)^{-2} \end{pmatrix} - \begin{pmatrix} (\lambda - p^2)^{-2} & 0 \\ 0 & (\lambda - p^2)^{-2} \end{pmatrix} \begin{pmatrix} p^2 & 0 \\ -p^k & p^2 \end{pmatrix}$$

one directly obtains the estimate, for $|\arg\lambda| > \varepsilon$,

$$\|(\lambda - a)^{-1}x\| \leq C_1(\varepsilon)|\lambda|^{-1}\|x\| + C_2(\varepsilon)|\lambda|^{-2}\|ax\|, \qquad x \in D(a).$$

It follows from item 4 that the operator $-a$ generates, on $D(a)$, an analytic semigroup of linear operators strongly continuous at zero. In ([2]), from which the example is taken, an explicit expression for this semigroup is also given (p. 198), and a wish is expressed for the construction of a theory of the abstract Cauchy problem for operators with unbounded resolvents.

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Note: Figure translations are in progress. See original paper for figures.