ON A CLASS OF HIGH-ORDER DIFFERENTIAL-OPERATOR EQUATIONS

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Abstract

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MATHEMATICS

Yu. A. DUBINSKII

ON A CLASS OF HIGH-ORDER DIFFERENTIAL-OPERATOR EQUATIONS

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This paper studies a certain class of differential-operator (d.-o.) equations (1), called the class of partially hyperbolic d.-o. equations.* The cases $s = 1, 2$ have been well studied; surveys of the main results and detailed bibliographies are available, for example, in ([1],[2]). Here we consider general boundary-value problems for d.-o. equations (1) of arbitrary order $s \geqslant 1$.

Let H be a complex separable Hilbert space, and let $u(t) : I \to H$ be a function of the real variable $t \in I \subset \mathbb{R}^1$, taking values in H. Let, further, $A : H \to H$ be a linear operator.

Consider the d.-o. equation

$$\mathfrak{A}(u) \equiv P_s\left(\frac{d}{dt}\right) u + Au = h(t), \tag{1}$$

where

$$P_s\left(\frac{d}{dt}\right) u \equiv \sum_{q=0}^{s} a_q u^{(q)}(t), \qquad a_q \in C^1, \qquad u^{(q)} \equiv d^q u/dt^q.$$

Definition. The operator $\mathfrak{A}(u)$ is called **partially hyperbolic** if the following conditions are satisfied: 1) if s is odd, then $a_s = i$ and the operator A is self-adjoint; 2) if $s = 2l$, then $a_s = (-1)^{l-1}$ and the operator A is self-adjoint and semibounded from below.

Notation. a) $H(l, 0; \gamma)$, $l \geqslant 0$, denotes the space of functions $u(t) : I \to H$ having finite norm

$$\|u\|^2_{l,0;\gamma} = \int_I \left(\|u^{(l)}\|^2 + \|u\|^2\right) e^{-\gamma t}\, dt,$$

where $\|u\| \equiv \|u(t)\|$ is the norm in H, and $\gamma \in \mathbb{R}^1$ is a parameter. b) $H(s, A; \gamma)$ denotes the space of functions $u(t) : I \to H$ having finite norm

$$\|u\|^2_{s,A;\gamma} \equiv \|u\|^2_{s,0;\gamma} + \|Au\|^2_{0,0;\gamma}.$$

c) If A is semibounded from below, then λ_0 denotes its first eigenvalue; if A is not bounded from below, then

$$\lambda_0 = \inf_{u \in D_A} \frac{|(Au, u)|}{\|u\|^2},$$

where D_A is the domain of definition of the operator Au.

I. The case of an operator A with point spectrum

§ 1. D.-o. equations on the entire axis (the case $I \equiv \mathbb{R}^1$)

Theorem 1. Let the operator $\mathfrak{A}(u)$ be partially hyperbolic. Then for every $\gamma \neq 0$ there exists a number $a(\gamma) \geqslant 0$ such that, for $\lambda_0 \geqslant a(\gamma)$, equation (1) has a unique solution $u(t) \in H(s, A; \gamma)$ for every right-

* Boundary-value problems for parabolic and elliptic equations of the form (1) are considered in ([3]).

part $h(t) \in H(1, 0; \gamma)$. In this case the estimate is valid

$$\|u\|_{s,A;\gamma} \leqslant K(\gamma)\|h\|_{1,0;\gamma}, \tag{2}$$

where $K(\gamma) > 0$ is a constant.

Corollary 1. If the operator $\mathfrak{A}(u)$ is partially hyperbolic, then equation (1) is normally solvable in the space $H(s, A; \gamma)$. This means that, for any function $h(t) \in H(1, 0; \gamma)$ subject to a finite number of conditions*, equation (1) is solvable in $H(s, A; \gamma)$. The homogeneous equation ($h(t) \equiv 0$) has a finite number of linearly independent solutions.

§ 2. Differential-operator equations on the half-axis (the case $I = R_+ \equiv [0, \infty)$).

Formulation of the problem. For a given function $h(t)$, it is required to find a solution of equation (1) satisfying, at $t = 0$, the conditions

$$R_\nu\left(\frac{d}{dt}\right)u\Big|_{t=0} = 0, \qquad \nu = 0, \dots, s-1-k, \tag{3}$$

where $k = [s/2]$ if s is odd, and $k = s/2 - 1$ if s is even (**problem I**); or $k = [s/2] + 1$ for any s (**problem II**).

Here

$$R_\nu\left(\frac{d}{dt}\right)u \equiv \sum_{q=0}^{n_\nu} p_{\nu q} u^{(q)}(t), \qquad p_{\nu q} \in C^1, \quad p_{\nu n_\nu} = 1, \quad n_\nu < s.$$

Suppose that the following conditions are satisfied.

Condition B. If $\lambda > 0$, then $\det \|\mu_j^{n_\nu}(\lambda)\| \neq 0$ $(\nu, j = 0, 1, \ldots, s-1-k)$, where $\mu_j(\lambda)$ are roots of the equation $a_s\mu^s + \lambda = 0$ such that $\operatorname{Re}\mu_j(\lambda) \leqslant 0$ (for problem I).

Condition C. If $\lambda > 0$, then $\det \|\mu_j^{n_\nu}(\lambda)\| \neq 0$ $(\nu, j = 0, \ldots, s-1-k)$, where $\mu_j(\lambda)$ are roots of the equation $a_s\mu^s + \lambda = 0$ such that $\operatorname{Re}\mu_j(\lambda) < 0$ (for problem II).

Theorem 2. Let $\gamma > 0$, let the operator $\mathfrak{A}(u)$ be partially hyperbolic, and let $\lambda_0 \geqslant 0$ be sufficiently large ($\lambda_0 \geqslant \alpha(\gamma)$). Then, if Condition B is satisfied, for any right-hand side $h(t) \in H(1, 0; \gamma)$ there exists a unique solution $u(t) \in H(s, A; \gamma)$ of problem I. In this case the estimate (2) is valid.

Theorem 3. Let $\gamma < 0$, let the operator $\mathfrak{A}(u)$ be partially hyperbolic, and let $\lambda_0 \geqslant \alpha(\gamma)$. Then, if Condition C is satisfied, for any right-hand side $h(t) \in H(1, 0; \gamma)$ there exists a unique solution $u(t) \in H(s, A; \gamma)$ of problem II. In this case the estimate (2) is valid.

Corollary 2. Let $\gamma > 0$, let the operator $\mathfrak{A}(u)$ be partially hyperbolic, and let Condition B be satisfied. Then problem I is normally solvable, i.e., has a finite-dimensional kernel and is solvable under a finite number of conditions on $h(t)$. In this case the inequality is valid

$$\|u\|_{s,A;\gamma} \leqslant K(\gamma)(\|h\|_{1,0;\gamma} + \|u\|_{0,0;\gamma}), \tag{4}$$

where $K(\gamma) > 0$ is a constant.

Corollary 3. Let $\gamma < 0$, let the operator $\mathfrak{A}(u)$ be partially hyperbolic, and let Condition C be satisfied. Then problem II is normally solvable and inequality (4) is valid.

If s is even, then the following holds.

Theorem 4. If, for the operator $\mathfrak{A}(u)$, where A is self-adjoint, problem I is normally solvable for every $\gamma > 0$, and problem II for every $\gamma < 0$, then the operator A is bounded below, i.e., the operator $\mathfrak{A}(u)$ is partially hyperbolic.

§ 3. Differential-operator equations on an interval (the case $I = [0, T]$).

In this case it is required to find a solution of equation (1) under the following condi–

* These conditions have the character of orthogonality conditions of $h(t)$ to a finite-dimensional subspace for all $\gamma \in R^1$, excluding a discrete set. This also applies to [3], where in Theorems 2.1 and 3.1 it is inaccurately stated that γ is arbitrary.

forms:

$$u(0) = 0, \dots, u^{(l)}(0) = 0, \quad u'(T) = 0, \dots, u^{(l-1)}(T) = 0, \quad s = 2l \vee 1. \qquad (5)$$

(here the second group of conditions in (5) is absent if $s = 1, 2$).

Theorem 5. Let the operator $\mathfrak{A}(u)$ be partially hyperbolic and let the number λ_0 be sufficiently large. Then for any function $h(t) \in H(1, 0; 0)$ the problem (1), (5) has a unique solution $u(t) \in H(s, A; 0)$. Moreover, the inequality

$$\|u\|_{s,A;0} \leq K\|h\|_{1,0;0}$$

holds, where $K > 0$ is a constant.

Corollary 4. If the operator $\mathfrak{A}(u)$ is partially hyperbolic, then the problem (1), (5) is normally solvable. Moreover, the inequality

$$\|u\|_{s,A;0} \leq K(\|h\|_{1,0;0} + \|u\|_{0,0;0})$$

holds, where $K > 0$ is a constant.

II. The case of a self-adjoint operator A with arbitrary spectrum.

Everywhere in what follows $s = 2l$ (and, consequently, $a_s = (-1)^{l-1}$) and the operator $A : H \to H$ is an arbitrary positive self-adjoint operator, i.e. $(Au, u) \geq \lambda_0(u, u)$, $\lambda_0 > 0$.

Consider the following polynomial in the real variable τ:

$$A(\gamma; \tau) \equiv \mathrm{Re}(\gamma/2 - i\tau)\,[P_s(\gamma/2 + i\tau) + \lambda_0].$$

Obviously, for $\gamma > 0$ the polynomial $A(\gamma; \tau)$ is semibounded below, and for $\gamma < 0$ it is semibounded above, i.e. there exists a constant $\alpha(\gamma) \geq 0$ such that, for $\gamma > 0$ ($\gamma < 0$), the polynomial

$$A_\lambda(\gamma; \tau) \equiv \mathrm{Re}(\gamma/2 - i\tau)\,[P_s(\gamma/2 + i\tau) + \lambda + \lambda_0] > 0 \quad (< 0)$$

for any $\lambda \geq \alpha(\gamma)$.

§ 1. D.-o. equations on the whole axis $\mathbb{R}^1$. Consider the equation

$$\mathfrak{A}_\lambda(u) \equiv \mathfrak{A}(u) + \lambda u = h(t), \qquad \lambda \geq \alpha(\gamma). \tag{6}$$

Theorem 6. Let the operator $\mathfrak{A}(u)$ be partially hyperbolic and $\gamma \neq 0$. Then for any right-hand side $h(t) \in H(1,0;\gamma)$ there exists a unique solution of equation (6), $u(t) \in H(s,A;\gamma)$. Moreover, the inequality

$$\|u\|_{s,A;\gamma} \leq K(\gamma)\|h\|_{1,0;\gamma} \tag{7}$$

holds, where $K(\gamma) > 0$ is a constant.

§ 2. D.-o. equations on a half-axis. Given a function $h(t)$, it is required to find a solution of equation (6) under the conditions

$$u(0) = 0, \dots, u^{(l)}(0) = 0 \quad \text{(problem I)},$$

or under the conditions

$$u'(0) = 0, \dots, u^{(l-1)}(0) = 0 \quad \text{(problem II)} \tag{8}$$

(for $s = 2$ the conditions (8) are absent).

Theorem 7. Let the operator $\mathfrak{A}(u)$ be partially hyperbolic and $\gamma > 0$ ($\gamma < 0$). Then for any function $h(t) \in H(1,0;\gamma)$ there exists a unique solution $u(t) \in H(s,A;\gamma)$ of problem I (problem II) for equation (6). Moreover, the inequality (7) holds.

§ 3. D.-o. equations on an interval. Given a function $h(t) \in H(1,0;0)$, it is required to find a solution of equation (6) on the interval $[0,T]$ under the conditions (5).

Theorem 8. If the operator $\mathfrak{A}(u)$ is partially hyperbolic, then for any right-hand side $h(t) \in H(1,0;0)$ problem (6), (5) has a unique solution $u(t) \in H(s,A;0)$. Moreover, estimate (7) is valid for $\gamma = 0$.

§ 4. **Examples.** Let $L(x,D)u$ be an elliptic self-adjoint operator, semibounded from below, in a domain $G \subset \mathbb{R}^n$, with a coercive system of boundary conditions. In the examples we restrict ourselves to the case of the cylinder $Q = \mathbb{R}^1_+ \times G$.

1. Consider an equation of Schrödinger-equation type

$$\pm iu' + L(x,D)u = h(t,x).$$

For $t = 0$, either the initial value $u(0,x) = 0$ is prescribed (the mixed problem), or there is no initial condition (the free problem). We have

$$A(\gamma;\tau) \equiv \gamma/2 + \lambda_0,$$

whence it follows that, for $\lambda_0 \geq 0$, the mixed (free) problem is uniquely solvable for any $\gamma > 0$ ($\gamma < 0$).

2. Consider an equation of wave-equation type

$$u'' + L(x, D)u = h(t, x),$$

for which the problems studied are either the classical mixed problem or the free problem. We have

$$A(\gamma; \tau) \equiv (\gamma/2)(\gamma^2/4 + \tau^2 + \lambda_0),$$

therefore, if $\lambda_0 \geq 0$, the mixed (free) problem is uniquely solvable for any $\gamma > 0$ ($\gamma < 0$). If $\lambda_0 < 0$, unique solvability holds for

$$\gamma > 2\sqrt{|\lambda_0|} \quad (\gamma < -2\sqrt{|\lambda_0|})$$

(cf. ([4],[5])).

3. Consider an equation of the form

$$-u^{\mathrm{IV}} + L(x, D)u = h(t, x) \tag{9}$$

under the conditions $u(0, x) = 0$, $u'(0, x) = 0$, $u''(0, x) = 0$ (problem I), or under the condition $u(0, x) = 0$ (problem II).

We have

$$A(\gamma; \tau) \equiv (\gamma/2)(3\tau^4 + \gamma^2\tau^2/2 - \gamma^4/16 + \lambda_0),$$

therefore, for $\lambda_0 > 0$, problem I (problem II) for equation (9) is uniquely solvable for

$$\gamma \in (0, 2\sqrt[4]{\lambda_0}) \quad (\gamma \in (-2\sqrt[4]{\lambda_0}, 0)).$$

For the remaining γ there is normal solvability. For $\lambda_0 \leq 0$, both problems are only normally solvable.

Remark. Let

$$P_s(\partial/\partial t, \partial/\partial x_1, \dots, \partial/\partial x_n)u = h(t, x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{10}$$

be a homogeneous strictly hyperbolic equation of order s. As is known (see ([6]–[8])), for equation (9) in Sobolev-Slobodetskii spaces with weight $e^{-\gamma t}$, for $\gamma > 0$, the Cauchy problem is correctly posed. If, however, equation (10) is considered in Sobolev-Slobodetskii spaces with weight $e^{-\gamma t}$ for $\gamma < 0$, then the problem without initial conditions is the correct one.

Moscow Power Engineering Institute

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Note: Figure translations are in progress. See original paper for figures.