ON REMOVABLE SINGULARITIES OF QUASICONFORMAL MAPPINGS IN SPACE

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Abstract

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ON REMOVABLE SINGULARITIES OF QUASICONFORMAL MAPPINGS IN SPACE

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1. Let E^n be n-dimensional Euclidean space; $|x|$ the length of the vector $x = (x_1, \dots, x_n) \in E^n$; G a domain, i.e., an open connected set; ∂G its boundary. Denote by $C_0^\infty(G)$ the class of infinitely differentiable finite functions $\varphi(x)$ with compact supports $\operatorname{supp}\varphi(x) \subset G$; by $W_n^1(G)$ the class of functions obtained as strong limits, in the norm

$$\|\varphi(x)\|_{W_n^1(G)} = \left[\int_G (|\varphi(x)|^n + |\nabla\varphi(x)|^n)\, dG\right]^{1/n},$$

of sequences of functions infinitely differentiable in $\overline{G}$.

We shall say that a function $\varphi(x)$ belongs to the class $\widetilde{W}_n^1(G)$ if $\varphi(x) \in W_n^1(G')$ for every strictly interior subdomain G', $\overline{G'} \subset G$.

Let $f(x) = (f_1(x), \dots, f_n(x))$ be a vector function defined almost everywhere in the domain G. We say that $f(x)$ belongs to the class $W_n^1(G)$ if each of its components $f_k(x) \in W_n^1(G)$ $(k = 1, \dots, n)$. Membership of a vector function in the class $\widetilde{W}_n^1(G)$ is defined analogously. Further, put

$$\lambda(x, f) = \left[\sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j}\right)^2\right]^{1/2}, \qquad I(x, f) = \det\left(\frac{\partial f_i}{\partial x_j}\right).$$

Following Yu. G. Reshetnyak ([1]), we shall call $f : G \to E^n$ a mapping with bounded distortion if $f \in \widetilde{W}_n^1(G)$ and there exists a constant $Q(f) \geq 1$ such that almost everywhere in G (in the sense of n-dimensional measure) the inequality

$$\lambda^n(x, f) \leq n^{n/2} Q(f) I(x, f). \tag{1}$$

is satisfied.

As shown in ([1]), every mapping with bounded distortion is continuous.

Homeomorphic mappings with bounded distortion are called quasiconformal mappings.

In this note we study sets of removable singularities for the indicated classes of mappings.

2. Let Δ be an open set in E^n; F its closed subset. For any $\alpha > 1$ define the α-capacity of the set F relative to Δ by

$$\operatorname*{cap}_{\alpha}(F, \Delta) = \inf \|\nabla\varphi(x)\|_{L^{\alpha}(\Delta)},$$

where the greatest lower bound is taken over all functions $\varphi(x)$ of the class $C_0^\infty(\Delta)$ such that $0 \le \varphi(x) \le 1$ for all $x \in \Delta$, $\varphi(x) = 1$ on F.

We shall be interested mainly in sets of zero α-capacity. Lemma 1 shows that this property of sets does not depend on the choice of the bounded set $\Delta \supset F$.

Lemma 1. Let $F \subset \Delta$, where Δ is an open bounded set in E^n, and let F be its closed subset such that

$$\operatorname{cap}_\alpha(F, \Delta) = 0.$$

Then

$$\operatorname{cap}_\alpha(F, \Delta_1) = 0$$

for every open bounded set $\Delta_1 \supset F$.

Let F be an arbitrary compact set in E^n. We shall say that F is a set of zero α-capacity and write

$$\operatorname{cap}_\alpha F = 0,$$

if, for at least one open bounded set $\Delta \supset F$,

$$\operatorname{cap}_\alpha(F, \Delta) = 0.$$

Lemma 2. Let F be a compact set in E^n of zero α-capacity ($\alpha \ge n$). Then every m-dimensional Hausdorff measure of the set F is equal to zero.

The proofs of Lemmas 1 and 2 are carried out analogously to the proofs of the corresponding assertions for n-capacity (see ([2])).

3. Let G be a domain in E^n, and let $F \subset G$ be a set closed relative to G which does not disconnect G, i.e., the open set $G \setminus F$ is connected. Denote by F_i the orthogonal projection of the set F onto the hyperplane

$$E_i^n = \{x = (x_1, \dots, x_n) \in E^n : \ x_i = 0\}.$$

We shall say that the set F projects onto a set of measure zero if each of its projections F_i ($i = 1, \dots, n$) has zero $(n-1)$-dimensional Lebesgue measure.

Lemma 3. If the vector-valued function $f \in W_n^1(G \setminus F)$ and the set $F \subset G$, closed relative to G, projects onto a set of measure zero, then

$$f \in W_n^1(G).$$

The proof of this assertion is based on the use of known properties of generalized derivatives (see ([3])).

Lemma 4. Let $f : G \to E^n$ be a mapping with bounded distortion. Then, for every compact set $F \subset G$, the inequality

$$\|\lambda(x, f)\|_{L^n(F)} \leq c(n)\, Q(f)\, \operatorname{cap}_\alpha(F, G)\, \|\, |f(x)|\, \|_{L^\beta(G \setminus F)} \tag{2}$$

holds, where $c(n)$ is an absolute constant; α, β are connected by the relation

$$1/\alpha + 1/\beta = 1/n.$$

Some special cases of this inequality were indicated by the author in the note ([4]).

The theorem formulated below is a generalization of the corresponding result of Yu. G. Reshetnyak ([5]).

Theorem 1. Let

$$f : (G \setminus F) \to E^n$$

be a mapping with bounded distortion, and let F be a set, closed relative to the domain G, of zero α-capacity ($\alpha \geq n$). If the mapping f belongs to the class $L^\beta(G \setminus F)$, where

$$1/\alpha + 1/\beta = 1/n,$$

then f can be extended to a mapping with bounded distortion on the whole domain G.

For the proof, on the basis of Lemma 4 we conclude that the mapping f belongs to the class $W_n^1(G' \setminus F)$ for any strictly interior subdomain G', $G' \Subset G$. Since

$$\operatorname{cap}_\alpha F = 0,$$

it follows, by Lemma 2, that the set F projects onto a set of measure zero. Taking Lemma 3 into account, we obtain

$$f \in W_n^1(G).$$

Relation (1) for the vector-valued function f is evidently satisfied almost everywhere in G, and the theorem is proved.

4. An analogous problem may also be considered for quasiconformal mappings in space. The first result in this direction—the removability of isolated singularities—belongs to B. V. Shabat ([6]). The removability of intervals and of some more complicated sets was proved by A. P. Kopylov and I. N. Pesin*. Theorem 2 formulated below is apparently stronger.

Theorem 2. Let

$$f : (G \setminus F) \to E^n$$

be a bounded quasiconformal mapping, and let F be a set closed relative to the domain G, pro–

* Report at the Siberian Colloquium on the Theory of Functions of a Complex Variable, 1968.

is mapped into a set of measure zero. Then f can be extended to a quasiconformal mapping on the whole domain G.

Let us give the idea of the proof.

Using Lemma 3, it is not difficult to establish that the given mapping is a mapping with bounded distortion in the whole domain G. Therefore f is continuous in G (see ([1])) and carries out an open, isolated mapping (see ([2,7])). In order that f be quasiconformal in G, it is necessary to prove its mutual single-valuedness. This can fail, obviously, only on the set F.

Suppose that there exist points $x_1, x_2 \in F$ for which $f(x_1) = f(x_2) = y_0$. Since the mapping is open in G, $y_0 \notin \partial f(G)$. From Theorem 3 of the paper ([8]) it follows that the n-dimensional measure of the set $f(F)$ is equal to zero, and any open ball $B(y_0, r)$ with center at y_0 and radius $r > 0$ contains points of the domain $f(G \setminus F)$.

Since the mapping is isolated, for sufficiently small $r_0 > 0$ there exist disjoint sets $\Delta(x_i)$, $x_i \in \Delta(x_i)$ $(i = 1, 2)$, for which

$$f(\Delta(x_i)) = B(y_0, r_0) \subset f(G).$$

It is not difficult to see that each of the $\Delta(x_i)$ has positive n-dimensional measure.

Let the point $y_1 \in B(y_0, r_0)$ be such that $y_1 \notin f(F)$. Then, by the preceding, there will be points $x', x'' \in (G \setminus F)$, $f(x') = f(x'') = y_1$, which contradicts the homeomorphism of f in the domain $G \setminus F$. The theorem is proved.

Remark. It is not difficult to see that for three-dimensional domains the removable ones include, in particular, all rectifiable curves. Of interest, in our view, is the circumstance that these curves may adjoin the boundary of the domain.

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CITED LITERATURE

1. Yu. G. Reshetnyak, *Sibirsk. matem. zhurn.*, **7**, No. 5, 1106 (1966).

2. Yu. G. Reshetnyak, ibid., **8**, No. 3, 629 (1967).

3. V. I. Smirnov, *A Course of Higher Mathematics*, 5, Moscow, 1959.

4. V. M. Miklyukov, *DAN*, **182**, No. 2 (1968).

5. Yu. G. Reshetnyak, International Congress of Mathematicians, Moscow, 1966, Abstracts of Short Scientific Communications, Section 9, Moscow, 1966, p. 42.

6. B. V. Shabat, *DAN*, **132**, No. 5, 1045 (1960).

7. Yu. G. Reshetnyak, *Sibirsk. matem. zhurn.*, **9**, No. 2, 368 (1968).

8. Yu. G. Reshetnyak, *Sibirsk. matem. zhurn.*, **7**, No. 4, 886 (1966).

Note: Figure translations are in progress. See original paper for figures.