ON THE CAUCHY PROBLEM FOR LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT

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Abstract

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Yu. G. BORISOVICH, A. S. TURBABIN

ON THE CAUCHY PROBLEM FOR LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT

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A number of works are devoted to the problem of studying solutions of differential equations with retarded argument by methods of functional analysis (see, for example, ($^{1-6}$)). The present article belongs to the same circle of questions. In it new formulas are obtained for the solution of the Cauchy problem for a linear nonhomogeneous system of differential equations with retarded argument. The cases of initial functions from the spaces L_p and C are considered. The use of the space L_p is caused not only by the desire to extend the class of initial functions (which is also of independent interest for the study of the shift operator in view of the special properties of the space L_p (see (^{6})), but also by the fact that the final formula is simple. The basic idea consists in passing to an "equivalent" differential equation without retardation in a Banach space with an unbounded operator. This approach gives more complete results than those of other authors (^{9}, ^{2}).

I. In this section the Cauchy problem is considered for functions from the space L_p. We describe the class of differential equations with retarded argument under consideration. Consider the space $M((-h,0),B)$ of abstract functions measurable on the interval $(-h,0)$ with values in a Banach space B. Let f be an additive and homogeneous operator with values in B and domain of definition $D(f) \subseteq M((-h,0),B)$. Let the restriction of f to the space of continuous functions $C((-h,0),B)$ be a linear bounded operator. Let $L_p([-h,T],B)$ $(T > 0,\ 1 \le p < \infty)$ be the space of abstract functions $x(\tau)$, defined on the interval $(-h,T)$ with values in B and summable to the p-th power, and let $x_t(s) = x(t+s)$, $-h \le s \le 0$, be a segment of the function $x(\tau)$; $x_t(s)$ is defined for $t \in [0,T]$. We impose additional conditions on the operator f: α) for any $T > 0$, for each function $x(\tau) \in L_p((-h,T),B)$, the function $a(t) = f(x_t(s))$ is defined for almost all $t \in [0,T]$ and is summable on $[0,T]$; β) the equivalence class of the function $a(t)$ depends only on the equivalence class of the function $x(\tau)$; γ) the inequality

$$\int_0^t |f(x_\tau(s))|_B \, d\tau \leq M_0(t)|x|_{L_p((-h,t),B)}$$

holds for any $t \in [0,T]$; here $|\cdot|_X$ denotes the norm of an element in the space X; $M_0(t)$ is a continuous function on $[0,+\infty)$.

Consider the Cauchy problem for the equation with retarded argument

$$\dot{x}(t) = f(x_t(s)) + \varphi(t), \qquad t \geq 0; \tag{1}$$

$$x(s) = x_0(s), \qquad -h < s < 0, \qquad x(0) = x_0, \tag{2}$$

where f is the operator described above; $\varphi(t) \in L_p((0,T),B)$ and $x_0 \in B$, $x_0(s) \in L_p((-h,0),B)$ are prescribed elements. The initial pair

We shall denote $\{x_0(s), x_0\}$ by $\tilde{x}_0(s)$. A solution of the Cauchy problem (1)–(2) is an abstract function $x(\tau)$, defined on $(-h,T]$, which is absolutely continuous for $t \geqslant 0$, satisfies equation (1) almost everywhere on $[0,T]$, coincides with $x_0(s)$ for $\tau \in (-h,0)$, and $x(0) = x_0$. It is not difficult to verify that, under conditions a)–c), the Cauchy problem has a unique solution corresponding to the initial condition $\tilde{x}_0(s)$, defined on $(-h,+\infty)$ and depending continuously on the initial conditions on each interval $[0,T]$. For the homogeneous equation

$$x'(t) = f(x_t(s)) \tag{3}$$

we define the shift operator along solutions by the equality

$$U(t)\tilde{x}_0(s) = \{x(t+s),\, x(t)\}, \quad t \geqslant 0. \tag{4}$$

Obviously, $U(t)$ is a strongly continuous semigroup of class C_0 in the space $L_p((-h,0),B) \oplus B$. Let A be the infinitesimal generator of the semigroup $U(t)$.

Lemma 1. *The infinitesimal operator A is the generator of the semigroup $U(t)$, and, moreover,*

a)

$$A\tilde{x}_0(s) = \{dx_0(s)/ds,\, f(x_0(s))\};$$

b) *the domain of definition*

$$D(A) = \{\tilde{x}_0(s) = \{x_0(s), x_0\} : x_0(s) \text{ is absolutely continuous, } x_0'(s) \in L_p((-h,0),B),\ x_0(0) = x_0\};$$

c) *$D(A)$ is everywhere dense in $L_p((-h,0),B) \oplus B$;*

d)

$$dU(t)\tilde{x}_0/dt = AU(t)\tilde{x}_0, \quad \tilde{x}_0 \in D(A);$$

e) *if the complex number* λ *is a regular point* (^{7}) *of the operator*

$$C_\lambda = f(e^{\lambda s} I \cdot) - \lambda I,$$

considered on B*, then it is a regular point of the operator* $A - \lambda I$*.*

Theorem 1. *Let the spectrum* $\sigma(A)$ *of the operator* A *be situated in the complex plane to the left of the line* $\operatorname{Re} z = \gamma$*. Then the solution* $x(t)$ *of equation (3) satisfies the estimate*

$$|\tilde{x}_t(s)|_{L_p((-h,0),B)\oplus B} \leqslant M_1 e^{(\gamma+\varepsilon)t} |\tilde{x}_0(s)|_{L_p((-h,0),B)\oplus B}, \tag{5}$$

where $\tilde{x}_0(s)$ *is the initial pair,* $\tilde{x}_t(s) = \{x(t+s), x(t)\}$*, and* $\varepsilon > 0$ *is arbitrary.*

We shall now derive a formula for the solution of the nonhomogeneous equation (1). To this end, consider the Cauchy problem for an ordinary differential equation in the Banach space $L_p \oplus B$

$$d\tilde{u}(t)/dt = A\tilde{u}(t) + \Phi(t), \tag{6}$$

$$\tilde{u}(0) = \tilde{u}_0(s) = \{u_0(s), u_0\}, \tag{7}$$

where A is the generator of the semigroup $U(t)$, and $\Phi(t) = \{0, \varphi(t)\}$ is a function with values in $L_p \oplus B$. By a solution of problem (6)–(7) we shall mean a continuous function $\tilde{u}(t)$, $0 \leqslant t \leqslant T$, which everywhere for $t > 0$ satisfies equation (6) and $\tilde{u}(0) = \tilde{u}_0(s)$.

Lemma 2. *Let* $\varphi(t) \in C([0,T], B)$*,* $\tilde{u}(0) \in D(A^2)$*, and let* $\tilde{u}(t) = \{u(t,s), u(t)\}$ *be a solution of problem (6)–(7); then the function*

$$x(t) = \begin{cases} u_0(t), & -h \leqslant t < 0, \\ u_0, & t = 0, \\ u(t), & t > 0, \end{cases}$$

will be a solution of problem (1)–(2). Conversely, if $x(t)$ *is a solution of problem (1)–(2) with initial pair* $\{x_0(s), x_0\} \in D(A^2)$*, then the function* $\tilde{u}(t) = \{x(t+s), x(t)\}$ *will be a solution of problem (6)–(7) with initial element* $\tilde{u}(0) = \{x_0(s), x_0\}$*.*

From Lemma 2 and known results of the theory of equations in Banach spaces (^{8}, p. 461), the following follows.

Theorem 2. *If $\varphi(t) \in C([0,T],B)$ and $\tilde{x}_0(s) = \{x_0(s), x_0\} \in L_p \oplus B$, then the solution $x(t)$ of problem (1)–(2) for $t \geqslant 0$ is the second component of the function*

$$\tilde{u}(t) = U(t)\tilde{x}_0(s) + \int_0^t U(t-\tau)\Phi(\tau)\,d\tau,$$

where $U(t)$ is the semigroup with infinitesimal generator A.

II. Consider the Cauchy problem with initial functions from the space $C = C([-h,0],B)$

$$x'(t) = f(x_t(s)) + \varphi(t), \qquad t \geqslant 0, \quad x(s) = x_0(s), \quad -h \leqslant s \leqslant 0. \tag{8}$$

Let f be a linear continuous operator defined on C with values in B, and let $\varphi(t) \in C([0,T],B)$ be a known function. A solution of this problem is a function $x(t)$, $-h \leqslant t \leqslant T$, satisfying equation (8) everywhere on $[0,T]$ and such that $x(t) = x_0(t)$ for $t \in [-h,0]$. The theorem on existence and uniqueness of the solution of problem (8), and questions connected with the semigroup for the homogeneous equation $x'(t) = f(x_t(s))$, have been studied earlier (see, for example, ([1,2,5])). We shall be interested in the formula for the solution of problem (8).

We note that a similar problem, at the initiative of one of the authors, was discussed several years ago at a seminar, where an important consideration was stated by Yu. S. Kolesov. Here we shall consider a more general case, using the considerations expressed there by one of the authors.

Consider the infinitesimal generator of the shift semigroup $U(t)x_0(s) = x(t+s)$, $-h \leqslant s \leqslant 0$, on the solutions of the homogeneous equation. It has the form

$$Ax_0(s) = x_0'(s), \qquad D(A) = \{x_0(s),\, x_0'(s) \in C,\; x_0'(0) = f(x_0(s))\}.$$

Lemma 3. If $x(t)$ is a solution of problem (8) with initial function $x_0(s)$ such that

$$x_0'(0) = f(x_0(s)) + \varphi(0), \qquad x_0'(s) \in C, \tag{*}$$

then the function $u(t) = x(t+s)$, $-h \leqslant s \leqslant 0$, for $t \geqslant 0$ satisfies the boundary-value problem

$$du/dt = \tilde{A}u(t), \qquad u(0) = x_0(s); \tag{9}$$

$$\partial u(t)/\partial s\big|_{s=0} = f(u(t)) + \varphi(t), \tag{10}$$

where $\tilde{A} = \partial/\partial s$ is the differentiation operator in the space C with domain $D(\tilde{A}) = C^1([-h,0],B)$.

Let $u(t)$ be a solution of problem (9)–(10); then $u(t) = x(t+s)$, and the function $x(t)$ $(-h \leqslant t \leqslant T)$ will be a solution of problem (8) with initial function $x_0(s) = u(0)$, satisfying $(*)$.

We shall seek the solution of problem (9)–(10) in the form $u(t) = z(t) + v(t)$. Suppose that it has been possible to find a function $v(t) = v(t,s)$ such that:

v_1) there exists $\partial v(t,s)/\partial t$ uniformly with respect to $s \in [-h,0]$, and it is continuous;

v_2) $v(t,s)$ is continuously differentiable with respect to s;

v_3) $\partial v(t,s)/\partial s\big|_{s=0} = f(v(t,s)) + \varphi(t)$;

then $z(t)$ will be a solution of the problem

$$dz/dt = Az(t) + \Phi(t), \qquad z(0) = x_0(s) - v(0,s), \tag{1}$$

$$\partial z/\partial s\big|_{s=0} = f(z(t)), \tag{11}$$

where $\Phi(t) = Av(t) - dv(t)/dt$.

Problem (11) can be interpreted as the Cauchy problem for an ordinary differential equation in a Banach space

$$dz(t)/dt = Az(t) + \Phi(t), \qquad z(0) = x_0(s) - v(0,s); \tag{12}$$

Here A is the infinitesimal generator of a strongly continuous semigroup for the homogeneous equation, and $z(0) \in D(A)$. The solution of problem (12) is given by the formula [8]

$$z(t) = U(t)z(0) + \int_0^t U(t-\tau)\Phi(\tau)\,d\tau. \tag{13}$$

Thus, it remains for us to prove the existence of the function $v(t,s)$. First consider the case when B is an n-dimensional Euclidean space. We solve the equation

$$\partial v(t,s)/\partial s = (I + \Lambda(s))f(v(t,s)) + \varphi(t+s), \tag{14}$$

where $\Lambda(s) = (-\delta_{ij}s^{\nu})_{i,j=1,2,\ldots,n}$, δ_{ij} is the Kronecker symbol, $\nu > 0$. Integrating (14), we obtain

$$v(t,s) = \tilde{\Lambda}(s)\cdot\psi(t) + \int_{t-h}^{t+s}\varphi(u)\,du, \tag{2}$$

$$\tilde{\Lambda}(s) = \left(\delta_{ij}\cdot\left[s+h-\frac{s^{\nu+1}+(-h)^{\nu+1}}{\nu+1}\right]\right), \qquad \psi(t) = f(v(t,s)). \tag{15}$$

Substituting the right-hand side of (15) into ψ, we obtain

$$\psi(t) = R(\nu)\psi(t) + f\left(\int_{\nu+1}^{t+s}\varphi(u)\,du\right), \tag{3}$$

$$R(\nu) = (a_{ij} - b_{ij}(\nu))_{i,j=1,\dots,n}, \qquad a_{ij} = f_i(0,\dots,s+h,\dots,0), \tag{16}$$

$$b_{ij}(\nu) = f_i\left(0,\dots,\frac{s^{\nu+1}+(-h)^{\nu+1}}{\nu+1},0,\dots,0\right), \tag{4}$$

where the nonzero element of the row stands in the j-th place.

System (16) is solvable, since $\det(I - R(\nu)) \neq 0$ for sufficiently small ν. Indeed:

1) if $\det(\delta_{ij} - a_{ij}) \neq 0$, then the matrix $\Lambda(s)$ in (14) must be taken identically equal to zero;
2) if $\det(\delta_{ij} - a_{ij}) = 0$, then, since $b_{ij}(\nu) \to a_{ij}$ as $\nu \to 0$, $\det(I - R(\nu)) \to 1$.

Remark. One should take ν in the form $2/(2k+1)$, so that the elements of the matrix $R(\nu)$ are real. In the general case of a Banach space B, the function $v(t,s)$ is constructed by the same scheme.

Voronezh State University

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Note: Figure translations are in progress. See original paper for figures.

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