INFINITE NONABELIAN GROUPS WITH THE MINIMALITY CONDITION FOR NONINVARIANT ABELIAN SUBGROUPS

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Abstract

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INFINITE NONABELIAN GROUPS WITH THE MINIMALITY CONDITION FOR NONINVARIANT ABELIAN SUBGROUPS

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Infinite groups with minimality conditions for one or another system of subgroups occupy a prominent place in the general theory of groups. For a survey of the principal results and problems concerning them (covering the twenty-year period 1939-1959), see (^1). Research in this direction has yielded a number of results determining conditions for the existence of infinite subgroups with certain properties in certain infinite groups. One such result is the author' s theorem on the extremality of an infinite locally soluble group satisfying the minimality condition for abelian subgroups (see (^2, ^3)). This theorem determines the condition for the existence, in an infinite locally soluble group, of an abelian subgroup having an infinite descending chain of subgroups; it follows from this theorem that such subgroups are absent from an infinite locally soluble group only when it is extremal, i.e., when it contains an abelian normal divisor of finite index that decomposes into a direct product of a finite number of quasicyclic groups.

The present work arose in connection with the question of conditions for the existence, in an infinite locally soluble group $\mathfrak{G}$, of an abelian subgroup having an infinite descending chain of subgroups noninvariant in $\mathfrak{G}$. The paper considers groups in which there exists no infinite descending chain

$$\mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_k \supset \cdots$$

of noninvariant abelian subgroups (the **minimality condition for noninvariant abelian subgroups**). The minimality condition for noninvariant abelian subgroups is, evidently, satisfied by all abelian groups, and also by all finite groups; in the present paper we consider infinite nonabelian groups satisfying this condition, J-groups. The class of J-groups contains, evidently, all infinite nonabelian groups satisfying the minimality condition for abelian subgroups, as well as all infinite nonabelian groups that do not have infinite noninvariant abelian subgroups and, in particular, the JH-groups considered by the author in the article (^4). A JH-group is an infinite nonabelian group containing an infinite abelian normal divisor and containing no infinite noninvariant abelian subgroups.

1. In the present section we consider nonperiodic J-groups. Let R be some element of infinite order of a J-group $\mathfrak{G}$. In view of the definition of a J-group, the sequence

$$\{R\} \supset \{R^p\} \supset \{R^{p^2}\} \supset \cdots \supset \{R^{p^k}\} \supset \cdots$$

for any natural number p must contain a normal divisor

$\{R^{p^n}\}$. Let p_1 and p_2 be two distinct prime numbers, let $\{R^{p_1^{n1}}\}$ and $\{R^{p_2^{n2}}\}$ be two corresponding normal divisors of this kind, and let a_1 and a_2 be two such integers that $a_1p_1^{n_1} + a_2p_2^{n_2} = 1$. Then

$$R = R^{a_1p_1^{n1}+a_2p_2^{n2}} = R^{a_1p_1^{n1}} \cdot R^{a_2p_2^{n2}},$$

$$X^{-1}RX = X^{-1}R^{a_1p_1^{n1}}X \cdot X^{-1}R^{a_2p_2^{n2}}X = \left(X^{-1}R^{p_1^{n1}}X\right)^{a_1}\left(X^{-1}R^{p_2^{n2}}X\right)^{a_2}$$

for any element X of $\mathfrak{G}$. In view of the invariance of the subgroups $\{R^{p_1^{n1}}\}$ and $\{R^{p_2^{n2}}\}$, it follows that $X^{-1}RX \in \{R\}$. Since X is an arbitrary element of the group $\mathfrak{G}$, this means that the subgroup $\{R\}$ is invariant in $\mathfrak{G}$. Thus, every element of infinite order of the J-group $\mathfrak{G}$ generates in it a cyclic normal divisor.

Let $\mathfrak{A}$ be the centralizer of the element R in the group $\mathfrak{G}$. Since the subgroup $\{R\}$ is invariant in $\mathfrak{G}$, the centralizer $\mathfrak{A}$ is invariant in $\mathfrak{G}$ and has in $\mathfrak{G}$ index not exceeding the number 2. Let A be an arbitrary element of $\mathfrak{A}$. If the element A has finite order, then RA is an element of infinite order and, hence, the subgroup $\{RA\}$ is invariant in $\mathfrak{G}$. But then for any $X \in \mathfrak{G}$ we have either $X^{-1}RAX = RA$, or $X^{-1}RAX = (RA)^{-1}$. If $A^n = 1$, then, using the commutativity of the elements R and A, and also the commutativity of the corresponding elements $X^{-1}RX$ and $X^{-1}AX$, we obtain from these relations, respectively, $X^{-1}R^{nX} = R^n$ and $X^{-1}R^{nX} = R^{-n}$. On the other hand, in view of the invariance of the subgroup $\{R\}$ in $\mathfrak{G}$, one of the two relations $X^{-1}RX = R$ or $X^{-1}RX = R^{-1}$ holds. Since R is an element of infinite order, it is then easily seen that the relation $X^{-1}RAX = RA$ is incompatible with the relation $X^{-1}RX = R^{-1}$, and the relation $X^{-1}RAX = (RA)^{-1}$ with the relation $X^{-1}RX = R$. Using this, we easily obtain that either $X^{-1}AX = A$, or $X^{-1}AX = A^{-1}$. But this means that the subgroup $\{A\}$ is invariant in the group $\mathfrak{G}$. If A is an element of infinite order, then the subgroup $\{A\}$ is invariant in $\mathfrak{G}$ by virtue of the preceding assertion. Thus, all cyclic subgroups of $\mathfrak{A}$ are invariant in $\mathfrak{G}$, and hence also in $\mathfrak{A}$. Since the subgroup $\mathfrak{A}$, obviously, contains elements of infinite order, it cannot be Hamiltonian (see (5)) and, consequently, is abelian.

Since the group $\mathfrak{G}$ is nonabelian (by the very definition of a J-group), it follows from this, in particular, that none of its elements of infinite order can belong to its center; therefore $\mathfrak{A} \neq \mathfrak{G}$, and hence the index of the subgroup $\mathfrak{A}$ in $\mathfrak{G}$ is equal to two.

Let B be an arbitrary element of $\mathfrak{G} - \mathfrak{A}$. Since the index of the subgroup $\mathfrak{A}$ in $\mathfrak{G}$ is equal to two, we have $B^2 \in \mathfrak{A}$. The order of the element B^2 cannot be infinite, because this element is, obviously, contained in the center of the group $\mathfrak{G}$. But then the element RB^2 has infinite order, where R is some element of infinite order from $\mathfrak{A}$. Since in $\mathfrak{G}$ every infinite cyclic subgroup is invariant, but is not contained in the center, the relations hold

$$B^{-1}(RB^2)B = (RB^2)^{-1} = B^{-2}R^{-1} = R^{-1}B^{-2},$$

$$B^{-1}(RB^2)B = B^{-1}RB \cdot B^{-1}B^2B = B^{-1}RB \cdot B^2 = R^{-1}B^2.$$

But then $R^{-1}B^{-2} = R^{-1}B^2$, and therefore $B^4 = 1$.

Let A be an arbitrary element of $\mathfrak{A}$. If it has infinite order, then from the fact that the subgroup $\{A\}$ is invariant in the group $\mathfrak{G}$, but is not contained in its center, the relation $B^{-1}AB = A^{-1}$ follows. If the element A has finite order, then RA is an element of infinite order and therefore

$$B^{-1}(RA)B = (RA)^{-1} = A^{-1}R^{-1} = R^{-1}A^{-1}.$$

However, on the other hand,

$$B^{-1}(RA)B = B^{-1}RB \cdot B^{-1}AB = R^{-1}B^{-1}AB.$$

Hence we obtain $B^{-1}AB = A^{-1}$. Thus, a J-group $\mathfrak{G}$ containing elements of infinite order has an abelian normal divisor $\mathfrak{A}$ of index 2 and an element B of order two or four such that $\mathfrak{G} = \mathfrak{A}\{B\}$ and $B^{-1}AB = A^{-1}$ for every $A \in \mathfrak{A}$. Hence, by virtue of the assertion formulated in paper (^{4}), it follows that

Theorem 1. *If a J-group contains elements of infinite order, then it is a JH-group.*

In addition we note that it necessarily contains nonidentity elements of finite order and has a finite center, which is an elementary abelian 2-group.

2. The present section is devoted to periodic J-groups. As already noted, the class of such groups contains, in particular, infinite nonabelian groups satisfying the minimality condition for abelian subgroups. Unfortunately, almost nothing is known in the general case about the structure of the latter. It is not even known whether they are locally finite. The author has put forward the hypothesis of the extremality of an arbitrary infinite locally finite group satisfying the minimality condition for abelian subgroups (see ([1])). So far it is known only that extremality holds here under the additional requirement of solvability of all finite subgroups (see ([2,3])). Therefore arbitrary J-groups satisfying the minimality condition for abelian subgroups are not considered in the present paper. For brevity, the periodic J-groups considered below that do not satisfy this condition will be called JL**-groups**.

Lemma 1. *If an abelian subgroup* $\mathfrak{A}$ *of a* J*-group* $\mathfrak{G}$ *contains an infinite set of elements of prime orders, then every cyclic subgroup of* $\mathfrak{A}$ *is invariant in* $\mathfrak{G}$*.*

Corollary. *If a* JL*-group has no quasicyclic subgroups, then it is a* JH*-group.*

Lemma 2. *Every* JL*-group is locally finite and locally solvable.*

Lemma 3. *Every* JL*-group* $\mathfrak{G}$ *has an infinite abelian normal divisor of finite index, all of whose cyclic subgroups are invariant in* $\mathfrak{G}$*.*

From this lemma, from Section 1, and from the extremality noted above of an infinite locally solvable group satisfying the minimality condition for abelian subgroups, the following assertion follows.

Theorem 2. *An infinite locally solvable group satisfying the minimality condition for noninvariant abelian subgroups has an infinite abelian normal divisor of finite index.*

Theorem 3. *A* JL*-group whose center does not satisfy the minimality condition for subgroups is a Hamiltonian group. Every* JL*-group whose center satisfies the minimality condition for subgroups has an abelian or Hamiltonian subgroup* $\mathfrak{N}$ *that does not satisfy the minimality condition for subgroups, all cyclic subgroups of which are invariant in* $\mathfrak{G}$*, and which determines a nontrivial cyclic factor group* $\mathfrak{G}/\mathfrak{N}$*.*

An infinite periodic group with center satisfying the minimality condition for subgroups is not always a JL-group if it has a normal divisor $\mathfrak{N}$ with the properties indicated in the theorem.

From Theorem 3 it follows that

Corollary. *If a* p*-group does not satisfy the minimality condition for abelian subgroups but satisfies the minimality condition for noninvariant abelian subgroups, then for* $p > 2$ *it is abelian, and for* $p = 2$ *it is abelian or Hamiltonian.*

3.

In conclusion we note some corollaries and strengthenings of the propositions formulated above. From Theorem 1 and Lemma 1 it follows:

Theorem 4. *If an infinite nonabelian group has a non-invariant abelian subgroup that does not satisfy the minimality condition for subgroups, then it also has an infinite descending chain of such subgroups.*

Indeed, from Theorem 1 and Lemma 1 it follows that every abelian subgroup of the J-group $\mathfrak{G}$ that does not satisfy the minimality condition for subgroups is invariant in $\mathfrak{G}$.

From the arguments given in item 1, it is not difficult to see the validity of the following strengthening of Theorem 1.

If an infinite nonabelian group containing elements of infinite order satisfies the minimality condition for non-invariant cyclic subgroups, then it is a JH-group.

Hence the following supplement to Theorem 4 follows.

If an infinite nonabelian group $\mathfrak{G}$, containing elements of infinite order, has an infinite non-invariant abelian subgroup, then it has an infinite descending chain of non-invariant cyclic subgroups.

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