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Abstract

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ON PRECOMPLETE CLASSES IN k-VALUED LOGICS

The note considers questions connected with the completeness problem for k-valued logics*.

1. As is well known (1), one of the ways of solving the completeness problem is to find all the so-called precomplete classes of a given finite-valued logic P_k. In 1921 there appeared a report on results of E. Post, which were published in detail only in 1941 (2). E. Post constructed, in particular, all precomplete classes for P_2. In 1954, in a work of S. V. Yablonskii (3), all precomplete classes in P_3 were described. At approximately the same time A. V. Kuznetsov pointed out the fundamental possibility of constructing all precomplete classes in P_k and established that the number $\pi(k)$ of these classes is not greater than 2^{k^k} (1). In connection with this, the question arose of an explicit description of all precomplete classes in P_k. Some families of precomplete classes were constructed in 1953 by A. V. Kuznetsov and S. V. Yablonskii and described in detail in (1). In 1959 V. V. Martynyuk (4), in 1962-1964 Lo Chzhu-kai, Pan Yun-tse, Wang Syang-hao, and Lo Sui-hua (5−10), and in 1965 E. Yu. Zakharova (11) constructed other families of precomplete classes. In 1965 a note by I. Rosenberg (12) appeared, in which it was reported that all precomplete classes in P_k are exhausted by 6 families, of which previously 4 were completely known and 2 partially (the proof of I. Rosenberg' s theorem has not yet been published).

At the same time the question arose of estimating the number of precomplete classes**. In 1952 A. V. Kuznetsov announced the result:
$\pi(k) \geq 2^{k(1-\varepsilon)}$, where $\varepsilon \to 0$ as $k \to \infty$, as was reported in (13). However, it subsequently became clear that A. V. Kuznetsov does not have a proof of this fact.

The present note is devoted to estimating the number of precomplete classes in P_k. The proofs of the assertions given rely on I. Rosenberg' s theorem, whose validity the authors know how to prove.

2. We introduce the following notation for the families of precomplete classes described in (^{12}): M is the family of classes of monotone functions, S of self-dual functions, L of quasilinear functions, U of functions preserving partitions of the set $E_k = \{0, 1, \dots, k-1\}$, C of functions preserving central predicates, B is the family of classes of functions that are homomorphic images of classes of functions preserving elementary predicates. Let us recall the definition of an h-place central predicate, which plays an important role in our arguments. Consider a completely symmetric predicate $\rho(y_1, \dots, y_h)$, defined on E_k and possessing the property of reflexivity—this means that ρ takes the value 1, in particular, on all tuples with repeated entries. A nonempty set $\mathfrak{C} \subset E_k$ is called a center of ρ if for any $c \in \mathfrak{C}$ and any $a_2, \dots, a_h \in E_k$, $\rho(c, a_2, \dots, a_h) = 1$. A predicate having a center is called central.

Theorem 1.

$$\pi(k) \sim \delta(k) k \, 2^{C_{k-1}{}^{[(k-1)/2]}},$$

where $\delta(k) = 1$, if k is odd; $\delta(k) = 2$, if k is even.

Proof. It can be shown that different precomplete classes are classes of preservation of different predicates; therefore the problem reduces to estimating the number of predicates for the families M—B. A rough count gives an upper estimate for $l(k)$—the number of predicates defining the families M, S, L, U, B—equal to 2^{k^2}. For $c(k)$—the number of central predicates—an asymptotic estimate is established. Comparison of the asymptotic estimate for $c(k)$ and the upper estimate for $l(k)$ shows that $\pi(k) \sim c(k)$.

Upper estimate. Every central h-ary predicate is completely determined by specifying a subset of the combinations of k elements taken h at a time that enter into its truth domain. For example, predicates whose center

Table 1*

k	M	S	L	U	C	B	Total number
2	11	11	11	00	21	00	54
3	31	11	11	31	93	11	188
4	182	11	11	133	407	72	8016
5	1904	241	121	505	35519	364	66734
6	337512	442	00	1909	11 49077	1717	15 237107

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* All concepts that are not defined here can be found in (^{1}).

** The problem was posed by S. V. Yablonskii in 1954 in lectures on many-valued logics at Moscow University.

k	M	S	L	U	C	B	Total number
7	93 68139	7201	3601	91713	7 758 233	81311	7 854 724> 1500

* In the upper part of a cell is written the number of classes; in the lower part, the number of types.

contain the element α, must contain C_{k-1}^{h-1} tuples; the remaining tuples may vary. The number of variable combinations is equal to C_{k-1}^{h}. Hence we find the number of h-ary predicates with center α, and, taking into account the ranges of variation of h and α, obtain

$$c(k) \leqslant k \sum_{h=1}^{k-1} 2^{C_{k-1}{}^{h}} \leqslant \delta(k)\, k 2^{C_{k-1}{}^{[(k-1)/2]}}.$$

Lower estimate. As noted, the number of h-ary predicates whose center contains the element α and, possibly, some further elements is equal to $2^{C_{k-1}{}^{h}}$. The number of h-ary predicates whose center contains α, β and, possibly, some elements is equal to $2C_{k-2}^{h}$. Hence the number of h-ary predicates whose center contains only α is not less than

$$\sum_{h=1}^{k-2} \left(2^{C_{k-1}{}^{h}} - (k-1)2^{C_{k-2}{}^{h}}\right) + 1.$$

Taking into account that α can take k values, we obtain that the number of all predicates whose center consists of one element is not less than

$$k \left[\sum_{h=1}^{k-2} \left(2^{C_{k-1}{}^{h}} - (k-1)2^{C_{k-2}{}^{h}}\right) + 1\right] \geq$$

$$\geq k \left[\delta(k) 2^{C_{k-1}{}^{[(k-1)/2]}} - (k-1)(k-2)2 \cdot 2^{C_{k-2}{}^{[(k-2)/2]}} + 1\right] \geq \delta(k) k \cdot 2^{C_{k-1}{}^{[(k-1)/2]}}.$$

Thus, "almost all" precomplete classes are determined by central predicates.

Denote by $\tau(k)$ the maximum number of pairwise nonisomorphic precomplete classes in P_k (i.e., the number of types of precomplete classes).

Theorem 2. $\tau(k) \sim \pi(k)/k!$.

The proof of this assertion is a further development of the idea of the proof of Theorem 1.

3. Let us estimate the effectiveness of completeness criteria formulated in terms of precomplete classes for small values of k. A detailed study of the families of precomplete classes makes it possible to compile Table 1. We note that all precomplete classes for $k = 4$ were known before the appearance of the work of I. Rosenberg.* It is seen from Table 1 that completeness criteria in terms of precomplete classes are practically acceptable for $k \leqslant 4$ and are hard to survey for $k \geqslant 5$, and in terms of types of precomplete classes, respectively, for $k \leqslant 6$ and $k \geqslant 7$. Let us observe that Theorems 1 and 2, already for $k = 8$, allow one to compute $\pi(k)$ and $\tau(k)$ with relative error 10^{-5}**.

In conclusion we give lists of predicates characterizing the types of precomplete classes for $k = 4, 5$. To write the predicates we use the logical operations $\vee$, &, $-$, the predicates $\leqslant$, $=$, $g_i(y) = \operatorname{sgn}|i - y|$ and the functions $x + y \pmod k$ (the sign & is sometimes omitted). For each type of precomplete classes we write a predicate defining one class of the type,

$k = 4$. **Family M.**

$$\rho_1(y_1y_2) = (y_1 \leqslant y_2), \quad \rho_2(y_1y_2) = (y_1 \leqslant y_2) \,\&\, (\bar{g}_1(y_1) \vee \bar{g}_2(y_2)),$$

Family S.

$$\rho_3(y_1y_2) = g_0(y_1)g_1(y_2) \vee g_0(y_2)g_1(y_1) \vee g_2(y_1)g_3(y_2) \vee g_2(y_2)g(y_1).$$

Family L.

$$\rho_4(y_1y_2y_3y_4) = (y_1 = y_2)(y_3 = y_4) \,\&\, (y_1 = y_3) \vee$$
$$\times (y_2 = y_4)(y_1 = y_4)(y_2 = y_3) \vee \ \&_{i<j} \overline{y_i = y_j}.$$

Family U.

$$\rho_5(y_1y_2) = g_0(y_1)g_0(y_2) \vee [g_1(y_1) \vee g_2(y_1) \vee g_3(y_1)][g_1(y_2) \vee g_2(y_2) \vee g_3(y_2)],$$

$$\rho_6(y_1y_2) = [g_0(y_1) \vee g_1(y_1)][g_0(y_2) \vee g_1(y_2)] \vee [g_2(y_1) \vee g_3(y_1)][g_2(y_2) \vee g_3(y_2)],$$
$$\rho_7(y_1y_2) = g_0(y_1)g_0(y_2) \vee g_1(y_1)g_1(y_2) \vee [g_2(y_1) \vee g_3(y_1)][g_2(y_2) \vee g_3(y_2)].$$

Family C.

$$\rho_8(y_1) = g_0(y_1), \quad \rho_9(y_1) = g_0(y_1) \vee g_1(y_1),$$
$$\rho_{10}(y_1) = g_0(y_1) \vee g_1(y_1) \vee g_2(y_1),$$
$$\rho_{11}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2),$$
$$\rho_{12}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2) \vee$$
$$\vee g_1(y_2)g_2(y_1),$$

$$\rho_{13}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1) \vee g_1(y_2),$$

$$\rho_{14}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1) \vee g_0(y_2) \vee$$
$$\vee\, g_0(y_3).$$

Family B**.**

$$\rho_{15}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee$$
$$\vee\, [g_2(y_1) \vee g_2(y_2) \vee g_2(y_3)][g_3(y_1) \vee g_3(y_2) \vee g_3(y_3)],$$
$$\rho_{16}(y_1y_2y_3y_4) = \bigvee_{1 \leqslant i < j \leqslant 4} (y_i = y_j).$$

$k = 5$. **Family** M**.**

$$\rho_1(y_1y_2) = (y_1 \leqslant y_2), \quad \rho_2(y_1y_2) = (y_1 \leqslant y_2)[\bar{g}_1(y_1) \vee \bar{g}_2(y_2)],$$
$$\rho_3(y_1y_2) = (y_1 \leqslant y_2)[\bar{g}_1(y_1) \vee \bar{g}_2(y_2)g_3(y_2)],$$
$$\rho_3(y_1y_2) = (y_1 \leqslant y_2)[\bar{g}_1(y_1) \vee \bar{g}_2(y_2)g_3(y_2)],$$
$$\rho_4(y_1y_2) = (y_1 \leqslant y_2)[\bar{g}_1(y_1)\bar{g}_2(y_2) \vee \bar{g}_1(y_1)g_3(y_2) \vee g_2(y_1)\bar{g}_3(y_2)].$$

Family S**.**

$$\rho_5(y_1y_2) = g_0(y_1)g_1(y_2) \vee g_1(y_1)g_2(y_2) \vee g_2(y_1)g_3(y_2) \vee$$
$$\vee\, g_3(y_1)g_4(y_2) \vee g_4(y_1)g_0(y_2).$$

Family L**.**

$$\rho_6(y_1y_2y_3) = (2y_1 = y_2 + y_3).$$

Family U**.**

$$\rho_7(y_1y_2) = g_0(y_1)g_0(y_2) \vee [g_1(y_1) \vee g_2(y_1) \vee$$
$$\vee\, g_3(y_1) \vee g_4(y_1)]\,\&\,[g_1(y_2) \vee g_2(y_2) \vee g_3(y_2) \vee g_4(y_2)].$$
$$\rho_8(y_1y_2) = [g_0(y_1) \vee g_1(y_1)][g_0(y_2) \vee g_1(y_2)] \vee$$
$$\vee\, [g_2(y_1) \vee g_3(y_1) \vee g_4(y_1)][g_2(y_2) \vee g_3(y_2) \vee g_4(y_2)],$$

$$\rho_9(y_1y_2) = g_0(y_1)g_0(y_2) \vee g_1(y_1)g_1(y_2) \vee$$
$$\vee\, [g_2(y_1) \vee g_3(y_1) \vee g_4(y_1)][g_2(y_2) \vee g_3(y_2) \vee g_4(y_2)],$$

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* In 1966 A. I. Mal' tsev informed one of the authors that he had a proof of completeness of the indicated list of precomplete classes for $k = 4$.

** The number of classes in C for $k = 8$ is 549.758.283.756.

$$\rho_{10}(y_1y_2) = g_0(y_1)g_0(y_2) \vee [g_1(y_1) \vee g_2(y_1)][g_1(y_2) \vee g_2(y_2)]\vee \\ \vee [g_3(y_1) \vee g_4(y_1)][g_3(y_2) \vee g_4(y_2)],$$

$$\rho_{11}(y_1y_2) = g_0(y_1)g_0(y_2) \vee g_1(y_1)g_1(y_2) \vee g_2(y_1)g_2(y_2)\vee \\ \vee [g_3(y_1) \vee g_4(y_1)][g_3(y_2) \vee g_4(y_2)].$$

Family C**.** $\rho_{12}(y_1) = g_0(y_1)$, $\rho_{13}(y_1) = g_0(y_1) \vee g_1(y_1)$,

$$\rho_{14}(y_1) = g_0(y_1) \vee g_1(y_1) \vee g_2(y_1),$$

$$\rho_{15}(y_1) = g_0(y_1) \vee g_1(y_1) \vee g_2(y_1) \vee g_3(y_1),$$

$$\rho_{16}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2),$$

$$\rho_{17}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2) \vee g_1(y_2)g_2(y_1),$$

$$\rho_{18}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2)\vee \\ \vee g_1(y_2)g_2(y_1) \vee g_3(y_1)g_4(y_2) \vee g_3(y_2)g_4(y_1),$$

$$\rho_{19}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2)\vee \\ \vee g_1(y_2)g_2(y_1) \vee g_1(y_1)g_3(y_2) \vee g_1(y_2)g_3(y_1),$$

$$\rho_{20}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2) \vee g_1(y_2)g_2(y_1)\vee \\ \vee g_1(y_1)g_4(y_2) \vee g_1(y_2)g_4(y_1) \vee g_2(y_1)g_3(y_2) \vee g_2(y_2)g_3(y_1),$$

$$\rho_{21}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1)g_2(y_2) \vee g_1(y_2)g_2(y_1)\vee \\ \vee g_2(y_1)g_3(y_2) \vee g_2(y_2)g_3(y_1) \vee g_1(y_1)g_3(y_2) \vee g_1(y_2)g_3(y_1),$$

$$\rho_{22}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee [g_1(y_1) \vee g_3(y_1)][g_2(y_2)\vee \\ \vee g_4(y_2)] \vee [g_1(y_2) \vee g_3(y_2)][g_2(y_1) \vee g_4(y_1)],$$

$$\rho_{23}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1) \vee g_1(y_2),$$

$$\rho_{24}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1) \vee g_1(y_2)\vee \\ \vee g_2(y_1)g_3(y_2) \vee g_2(y_2)g_3(y_1),$$

$$\rho_{25}(y_1y_2) = (y_1 = y_2) \vee g_0(y_1) \vee g_0(y_2) \vee g_1(y_1) \vee g_1(y_2) \vee \\ \vee g_2(y_1) \vee g_2(y_2),$$

$$\rho_{26}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1) \vee \\ \vee g_0(y_2) \vee g_0(y_3),$$

$$\rho_{27}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1) \vee g_0(y_2) \vee \\ \vee g_0(y_3) \vee \bigwedge_{i=1}^{3} \bigvee_{j=1}^{3} g_i(y_j),$$

$$\rho_{28}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1) \vee g_0(y_2) \vee \\ \vee g_0(y_3) \vee \bigwedge_{i=1}^{3} \bigvee_{j=1}^{3} g_i(y_j) \vee \bigwedge_{i=1,\,2,\,4} \bigvee_{j=1}^{3} g_i(y_j),$$

$$\rho_{29}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee g_0(y_1) \vee \\ \vee g_0(y_2) \vee g_0(y_3) \vee g_1(y_1) \vee g_1(y_2) \vee g_1(y_3),$$

$$\rho_{30}(y_1y_2y_3y_4) = \bigvee_{i \neq j} (y_i = y_j) \vee g_0(y_1) \vee g_0(y_2) \vee g_0(y_3) \vee g_0(y_4).$$

Family B.

$$\rho_{31}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee \\ \vee \bigvee_{i \neq j} [g_2(y_i) \vee g_3(y_i) \vee g_4(y_i)][g_2(y_j) \vee g_3(y_j) \vee g_4(y_j)],$$

$$\rho_{32}(y_1y_2y_3) = (y_1 = y_2) \vee (y_1 = y_3) \vee (y_2 = y_3) \vee \\ \vee \bigvee_{i \neq j} [g_1(y_i) g_2(y_j) \vee g_3(y_i) g_4(y_j)],$$

$$\rho_{33}(y_1y_2y_3y_4) = \bigvee_{i \neq j} (y_i = y_j) \vee \bigvee_{i \neq j} g_3(y_i) g_4(y_j), \quad \rho_{34}(y_1y_2y_3y_4y_5) = \bigvee_{i \neq j} (y_i = y_j).$$

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REFERENCES

1. S. V. Yablonskii, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **51**, 5 (1958).
2. E. Post, Two-Valued Iterative Systems, 1941.
3. S. V. Yablonskii, DAN, **95**, No. 6, 1153 (1954).
4. V. V. Martynyuk, in: Problems of Cybernetics, issue 3, 1960, p. 49.
5. Lo Zhu-kai, Acta sci. natur. Univ. Jilinensis, No. 3 (1964).
6. Pan Yun-tse, ibid., No. 2 (1962).
7. Wang Xian-hao, ibid., No. 2 (1963).
8. Lo Zhu-kai, ibid., No. 2 (1) (1963).
9. Lo Zhu-kai, ibid., No. 2 (2) (1963).
10. Lo Zhu-kai, Lo Sui-hua, ibid., No. 4 (1963).
11. E. Yu. Zakharova, in: Problems of Cybernetics, issue 18, 1966, p. 5.
12. J. Rosenberg, C. R., **260**, Groupe 1, 5 IV (1965).
13. Mathematics over 40 Years, Moscow, 1959, p. 102.

Note: Figure translations are in progress. See original paper for figures.