D. G. SANIKIDZE

Let an infinite triangular matrix of nodes be given

1969

SovietRxiv

Abstract

Full Text

MATHEMATICS

D. G. SANIKIDZE

ON THE DIVERGENCE OF INTERPOLATION PROCESSES

(Presented by Academician P. S. Aleksandrov, 30 I 1969)

Let an infinite triangular matrix of nodes be given

$$-1 \le x_1^{(n)} < x_2^{(n)} < \dots < x_n^{(n)} \le 1 \quad (n = 1, 2, \dots). \tag{1}$$

For every function $f(x)$ defined on $[-1, +1]$, one can construct the Lagrange polynomial interpolating $f(x)$ at the nodes of the n-th $(n = 1, 2, \dots)$ row of matrix (1):

$$L_{n-1}(f; x) = \sum_{k=1}^{n} l_{n,k}(x) f(x_k^{(n)}),$$

$$l_{n,k}(x) = \frac{\omega_n(x)}{(x - x_k^{(n)})\omega_n'(x_k^{(n)})}, \qquad \omega_n(x) = \prod_{k=1}^{n} (x - x_k^{(n)}).$$

As is known, the sums

$$\sum_{k=1}^{n} |l_{n,k}(x)| \quad (n = 1, 2, \dots)$$

grow without bound for any system of nodes, and it is impossible to specify such a matrix (1) that, for every continuous function,

$$L_{n-1}(f; x) \to f(x) \quad \text{as } n \to \infty. \tag{2}$$

Meanwhile, as was shown by V. I. Krylov ([1]), if the matrix (1) is Chebyshev, then (2) holds uniformly on $[-1, +1]$ for every function f absolutely continuous on $[-1, +1]$. D. L. Berman ([2]) extended V. I. Krylov' s results to a sufficiently broad class of matrices (1), of which the Chebyshev system of nodes is a special case.

In this connection, the following question is of some interest: will a positive result hold in the class ACG_* (([3], p. 333) of functions?

The answer to this question is negative.

Theorem. *There exists no system of nodes guaranteeing the convergence of interpolation for all functions* $\in ACG_*$.

Proof. Following ([4]), for every $f \in ACG_*$ the interpolation remainder can be represented in the form

$$R_n(f;x) = \int_{-1}^{+1} \left[E(x-t) - \sum_{k=1}^{n} l_{n,k}(x) E(x_k^{(n)} - t) \right] f'(t)\, dt,$$

$$E(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

where the integral is understood in the Denjoy–Perron sense.

The further arguments are based on the following theorem of A. G. Djvaršeišvili ([5]).

Let $\{g_n(x)\}$ be a sequence of functions locally monotone* on $[-1,+1]$ such that

$$\overline{\lim}_{n\to\infty} \left| \int_{-1}^{+1} g_n(x)\psi(x)\, dx \right| < L \qquad (n = 1, 2, \dots) \tag{3}$$

for every summable function $\psi(x)$ on $[-1,+1]$. Then the inequalities

$$\left| \int_{-1}^{+1} g_n(x)\varphi(x)\, dx \right| \leq M(\varphi) \qquad (n = 1, 2, \dots)$$

can hold for every function $\varphi(x)$ integrable in the Denjoy–Perron sense if and only if the total variations of the functions $g_n(x)$ $(n = 1, 2, \dots)$ are bounded in the aggregate:

$$\operatorname*{Var}_{-1}^{+1} g_n(x) \leq N \qquad (n = 1, 2, \dots). \tag{4}$$

Since, for arbitrary x and n, the expression

$$F_n(t) = E(x-t) - \sum_{k=1}^{n} l_{n,k}(x) E(x_k^{(n)} - t)$$

is a piecewise constant function on $[-1, +1]$, it satisfies the condition of local monotonicity for every system of nodes (1). Moreover, as was noted above, there exist matrices of nodes such that

$$\lim_{n\to\infty} R_n(f;x) = 0$$

uniformly on $[-1, +1]$ for every absolutely continuous function f. Therefore, for such matrices of nodes, conditions (3) for the integrals

$$\int_{-1}^{+1} F_n(t) f'(t)\, dt \qquad (n = 1, 2, \ldots)$$

are satisfied. On the other hand, for any x,

$$\operatorname*{Var}_{-1}^{+1} \sum_{k=1}^{n} l_{n,k}(x) E(x_k^{(n)} - t) = \sum_{k=1}^{n} |l_{n,k}(x)|,$$

whence it follows that conditions (4) for the sequence $\{F_n(t)\}$ cannot be satisfied, whatever the matrix of nodes (1) may be. In view of this, according to the theorem of A. G. Dzhvarsheishvili, there exists a function $f \in ACG_*$ such that the sequence $\{R_n(f;x)\}$ will not be bounded. This proves the theorem.

Computing CenterAcademy of Sciences of the Georgian SSRTbilisi

Received28 I 1969

REFERENCES

[1] V. I. Krylov, DAN, 107, No. 3 (1956).
[2] D. L. Berman, DAN, 112, No. 1 (1957).
[3] S. Saks, Theory of the Integral, Moscow, 1949.
[4] V. I. Krylov, DAN, 105, No. 2 (1955).
[5] A. G. Dzhvarsheishvili, Communications of the Academy of Sciences of the Georgian SSR, 17, No. 4 (1956).

[*] For each point $x \in [-1, +1]$ there exist intervals $(x - \delta, x)$ and $(x, x + \delta)$ on which $g_n(x)$ is monotone.

Note: Figure translations are in progress. See original paper for figures.