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Abstract

Full Text

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MATHEMATICS

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ON THE PROBLEM OF ARITHMETICITY OF DISCRETE GROUPS

(Presented by Academician A. N. Kolmogorov, 3 XII 1968)

I. Definitions. A **discrete subgroup** of a topological group is a subgroup that intersects some neighborhood of the identity only in the identity.

An **algebraic group** is a subgroup of the group $GL(n, \mathbb{C})$ singled out by the conditions that a certain number of polynomials in the matrix entries vanish. If these polynomials can be chosen so that all their coefficients belong to the field of rational numbers \mathbb{Q} , then one says that this group is **defined over the field** \mathbb{Q} . If k is some ring, then G_k denotes the set of points of the group G defined over k , whose determinant is the identity of the ring k . If $G_R = G \cap GL(n, \mathbb{R})$, then G_Z (\mathbb{Z} is the ring of integers) is called **arithmetic**. Suppose the group G_R is the direct product of a compact group k and some group G_1 , and φ is the projection of G onto G_1 . It is easy to see that $\varphi(G_Z)$ is a discrete subgroup of the group G_1 . Discrete groups obtained by such a construction are also called **arithmetic groups**. Two subgroups are called **commensurable** if their intersection has finite index in each of them. A subgroup commensurable with an arithmetic subgroup is also called **arithmetic**.

An element $u \in G$ is called **unipotent** if all its eigenvalues are equal to 1. A subgroup of a group is called **unipotent** if all its elements are unipotent. The **unipotent radical** of a group $H \subset G$ is the maximal unipotent subgroup $U \subset H \subset G$ that is a normal divisor in H . We shall say that a closed unipotent subgroup is **horospherical** if it coincides with the unipotent radical of its normalizer. A closed subgroup P of the group G is called **parabolic** if the quotient space G/P is compact. It is not difficult to verify that a unipotent subgroup is horospherical if and only if its normalizer is a parabolic subgroup.

II. If G is a semisimple group and Γ is its arithmetic subgroup, then it is known that the volume of the quotient space G/Γ is finite. A. Selberg conjectured that, for almost all semisimple real Lie groups, all their discrete subgroups with quotient space of finite volume are arithmetic.

E. B. Vinberg and V. S. Makarov constructed a series of examples showing that this conjecture, in the case when the group G is the group of motions of Lobachevsky space (in dimensions 3, 4, and 5), is false.

Let us divide all discrete subgroups with quotient space of finite volume into two classes: 1) the quotient space G/Γ is compact; 2) the quotient space G/Γ is noncompact.

In Section III a plan of proof of the following theorem will be outlined:

Main theorem. *Let G be a semisimple real Lie group without compact factors, $\text{rank } G \geq 2$, Γ its discrete subgroup-*

* $\text{rank } G$ is the dimension of a maximal split torus. A split torus is a subgroup consisting of semisimple elements all of whose eigenvalues are real.

is finite, G/Γ is noncompact and the pair (G, Γ) is irreducible*. Then there exist a group G_Q and a parabolic subgroup $P \subset G$ such that $G_R = G$, $\Gamma \cap G_Q$ is commensurable with Γ , and $\Gamma \cap P$ and $G_Z \cap P$ are commensurable.

III. Let G be a connected Lie group. Then G/R , where R is the maximal solvable divisor in G , is a semisimple algebraic group.

Denote by f the natural mapping $G \rightarrow G/R$.

Theorem 1. *If Γ is a discrete subgroup in G and $f(\Gamma)$ is dense in the Zariski topology in G/R , then $f(\Gamma)$ is discrete.*

The proof of Theorem 1 is in many respects analogous to the proof of Bieberbach's classical theorem on discrete subgroups of the group of motions of n -dimensional Euclidean space.

Introduce on G a right-invariant measure which naturally induces on G/Γ a certain measure μ .

Corollary. *If*

$$\int_{G/\Gamma} d\mu < \infty,$$

then $f(\Gamma)$ is discrete.

Let T^t be a one-parameter group consisting of unipotent linear transformations of Euclidean space \mathbb{R}^n . The group T^t induces a group of transformations \tilde{T}^t in the space of lattices of full dimension in \mathbb{R}^{n**} . Consider in \mathbb{R}^n a lattice \mathbb{Z}^n of full dimension such that the unit neighborhood of 0 in \mathbb{R}^n contains no nonzero point of the lattice \mathbb{Z}^n .

Theorem 2. *There exists a $\delta > 0$ such that, for any $t_0 > 0$, there is a $t > t_0$ for which the intersection of the δ -neighborhood of 0 in \mathbb{R}^n with $\tilde{T}^t \mathbb{Z}^n$ consists only of 0.*

By Mahler's compactness criterion, Theorem 2 can be reformulated as follows:

Theorem 2'. *If the trajectory of a point $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ under the action of a one-parameter unipotent group $U \in SL(n, \mathbb{R})$ is closed, then it is compact.*

Corollary. *Let H^{***} be a semisimple algebraic subgroup of the group $SL(n, \mathbb{R})$, and let x be some point in the space $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$. Then, if the trajectory Hx of the point x under the action of the group H is closed, it has finite volume (with respect to the natural measure).*

In the proof of the corollary one uses the theorem that every affine algebraic variety in which the set of rational points is dense in the Zariski topology is defined over \mathbb{Q} .

In what follows in this section it will be assumed that G is a semisimple algebraic real Lie group without compact factors, Γ is its discrete subgroup,

$$\int_{G/\Gamma} d\mu < \infty,$$

G/Γ is noncompact, and the pair (G, Γ) is irreducible.

Consider a maximal unipotent subgroup U_Γ in the discrete group Γ , and take a minimal connected unipotent subgroup U_G containing U_Γ .

Using Theorems 1 and 2 and the methods of [5], Theorems 3 and 4 are proved.

Theorem 3. *U_G is a horospherical subgroup of the group G .*

Theorem 4. *There exists a parabolic subgroup $P \subset G$ such that $P \cap \Gamma$ is an arithmetic subgroup in P .*

Suppose now that $\text{rank } G \geq 2$.

* We shall call the pair (G, Γ) irreducible if, for any decomposition of G into a nontrivial direct product of Lie groups $G = G_1 \times G_2$, the product $(G_1 \cap \Gamma) \times (G_2 \cap \Gamma)$ is incommensurable with Γ .

** The topology in the space of lattices is natural.

*** In fact, it suffices to assume that H is an algebraic group having no nontrivial multiplicative characters.

Let U_P denote the unipotent radical of the group P , and let g_P be the natural mapping $P \rightarrow P/U_P$. Consider in G two parabolic subgroups P_1 and P_2 satisfying the conclusion of Theorem 4. Then the following holds.

Theorem 5. *$g_{P_1}(P_1 \cap P_2 \cap \Gamma)$ is commensurable with $g_{P_1}(P_1 \cap \Gamma)$.*

One can prove the following algebraic theorem.

Theorem 6. *Let P be a parabolic subgroup of the group G , defined over \mathbb{Q} . Then there exists a unique group G_Q such that $G_R = G$ and $G_Q \cap P = P_Q$.*

With the aid of Theorems 4, 5, and 6 the main theorem is proved.

IV. A fundamental domain of a discrete group Γ is a set which intersects each set of the form Γg (the trajectory of the element g under the action of the group Γ) in exactly one point. If O is a fundamental domain, then γO , where $\gamma \in \Gamma$, is also a fundamental domain. The number of sides of a fundamental domain is the number of those elements $\gamma \in \Gamma$ for which γO borders on O (i.e., the closure of γO intersects the closure of O).

From Theorem 4 it follows that

Theorem 7. If $\int_{G/\Gamma} d\mu < \infty$, then for Γ there exists a fundamental domain with a finite number of sides.

Theorem 7 admits the following refinement:

Theorem 7'. There exists a constant c , depending only on the group G , such that, if $\int_{G/\Gamma} d\mu < V$, then there exists a fundamental domain with a number of sides less than cV .

Using the methods of the works ⁽¹⁻⁴⁾ and Theorems 4-6 and the main theorem, one proves

Theorem 8. Let $\text{rank } G \geq 2$, $\Gamma \subset G$, $\int_{G/\Gamma} d\mu < \infty$, the pair (G, Γ) be irreducible, and let T be a finite-dimensional representation of the group G . Then any finite-dimensional representation T' of the group Γ , close to the restriction of T to Γ , is equivalent to it.

Theorem 8 proves Selberg's conjecture on the rigidity of finite-dimensional representations of discrete groups.

The following is related to Theorem 7.

Theorem 9. For a given group G there exists only a finite number of nondiffeomorphic manifolds of the form G/Γ whose volume is less than some previously given constant V .

From Mostow's theorem ⁽⁶⁾ and Theorem 9 one derives

Corollary. If $n \geq 3$, then there exists only a finite number of nonisometric manifolds of constant negative curvature whose volume is less than some previously given constant V .

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