TWO PROPOSITIONS ON BAER GROUPS

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Abstract

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TWO PROPOSITIONS ON BAER GROUPS

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Let there be given a certain operator U, assigning to every class of groups $\mathfrak{X}$ a definite class $U\mathfrak{X}$. The powers of the operator U are defined as follows: for any class of groups $\mathfrak{X}$ we put $U^0\mathfrak{X} = \mathfrak{X}$ and suppose that for all transfinite ordinal numbers β less than α, the operators U^β have already been defined. If α is a limit ordinal, put

$$U^\alpha\mathfrak{X} = \bigcup_{\beta<\alpha} U^\beta\mathfrak{X},$$

whereas if $\alpha = \gamma + 1$, then $U^\alpha\mathfrak{X} = U(U^\gamma\mathfrak{X})$. In addition, let

$$\overline{U}\mathfrak{X} = \bigcup_{\alpha} U^\alpha\mathfrak{X}.$$

The operator $\overline{U}$ is called the closure of the operator U.

For an arbitrary class of groups $\mathfrak{X}$, denote by $R\mathfrak{X}$ the class of groups generated by their normal divisors from $\mathfrak{X}$, by $K\mathfrak{X}$ the class of groups covered by their normal divisors from $\mathfrak{X}$, and by $F\mathfrak{X}$ the class of groups in which every finite set of elements is contained in a normal divisor from the class $\mathfrak{X}$.

Let $\mathfrak{A}$ denote the class of abelian groups, $\mathfrak{N}_f$ the class of finitely generated nilpotent groups, $\mathfrak{N}$ the class of nilpotent groups, $\mathfrak{B}$ the class of Baer groups, and F the class of finite groups. Groups from the class $R\mathfrak{N}$ are called fitting groups.

1. As is known, $\overline{R}\mathfrak{A} = \overline{R}\mathfrak{N} = \overline{F}\mathfrak{N}_f = \mathfrak{B}$. Recently R. S. Dark in (^{1}) constructed an example of a primary and, consequently, non-fitting Baer group, thereby showing that the classes $R\mathfrak{N}$ and $\overline{R}\mathfrak{N}$ are distinct. More complete information is given by the following theorem.

Theorem 1. *For every integer $n \geqslant 1$ there exists a Baer group which is not generated by its n-generated nilpotent subgroups.*

Proof. Fix a prime number $p \geqslant n+1$. Let H be the discrete direct product of a countable set of cyclic groups of order p. Construct a sequence of groups by putting

$$G_0 = H, \qquad G_i = \{a_i\} \operatorname{wr} G_{i-1}, \quad \text{where } a_i^p = 1, \quad i = 1, 2, \ldots$$

By successive application of the following known lemma it is proved that each G_i is a Baer group.

Lemma 1. *If a normal divisor B_0 of a group B is an abelian p-group of finite exponent, and B/B_0 is a Baer p-group, then B is also a Baer p-group.*

In the wreath product $A \operatorname{wr} B$ of two groups A and B, the copy of the group A corresponding to an element $b \in B$ is denoted by A_b, and the element of the group A_b corresponding to an element $a \in A$ is denoted by a_b.

Lemma 2. *Let $G = A \operatorname{wr} B$, where B is a p-group. Then for any elements $g, f \in B$, $g \neq 1$, $a \in A$, with $m < p$,*

$$[a_f, g(m)] = \prod_{i=0}^{m} a_{fg^i}^{r_i} \neq 1, \qquad \text{where } r_i = (-1)^{m-i}\binom{m}{i}.$$

In (^{2}), p. 394, the following assertion is given.

Lemma 3. *Let $G = A \operatorname{wr} B$, $g \in A^B$, $g \neq 1$, and let $\mathfrak{Z}(g)$ be the centralizer of the element g. Then $\mathfrak{Z}(g) \cap B$ is a finite group.*

We shall show that the group $G_{n+1} = \{a_{n+1}\}^{G_n} \lambda \ldots \lambda \{a_1\}^{G_0} \lambda H$ is not generated by its n-generated nilpotent subgroups. Suppose the contrary. Consider the normal divisors

$$H_m = \{a_{n+1}\}^{G_n} \lambda \ldots \lambda \{a_{n-m+2}\}^{G_{n-m+1}}, \qquad 1 \leq m \leq n+1,$$

and also let $H_0 = E$. In G_{n+1} there is a normal series

$$E \subset F_1 \subset F_2 \subset \ldots \subset F_{n-1} \subset F_n \subset G,$$

where F_1 is a nilpotent subgroup not lying in H_{n+1}. Take an element $g \in F_1 \setminus H_{n+1}$, and let $g_i = [b_i, g(n)]$, $i = 1, 2, \ldots, n+1$, where b_i is an arbitrary representative of the element a_i in the group $\{a_i\}^{G_{i-1}}$. From Lemma 2 it follows that $g_i \in H_{n-i+2} \setminus H_{n-i+1}$, $1 \leq i \leq n+1$. Moreover, since F_1 is of class n, all $g_i \in F_1$.

Construct the subgroups $M_1 = \{h^{-1} g_1 h \mid h \in H\}$, $M_2 = \{h^{-1} g_2 h \mid h \in M_1\}, \ldots,$ $\ldots, M_n = \{h^{-1} g_{nh} \mid h \in M_{n-1}\}$. By construction, $M_i \subset H_{n-i+2}$, $1 \leq i \leq n$. The subgroup M_1 lies in F_n, since it is generated by elements conjugate to $g_1 \in F_1 \subset F_n$. Next, M_2 lies in F_{n-1}, since it is generated by elements conjugate to $g_2 \in F_1 \subset F_{n-1}$ by means of elements from $M_1 \subset F_n$. Continuing, we obtain that M_n lies in F_1.

If C is a subgroup of the semidirect product $A \lambda B$, then the image of C under the homomorphism $\mu : A\lambda B \to B$ will be called the projection of C onto B.

Consider the projection of M_1 onto $\{a_1\}^{G_0}$. Let $g_1 = h_2 h_1$, $h_1 \in \{a_1\}^{G_0}$, $h_2 \in H_n$. Since $g_1 \in H_{n+1} \setminus H_n$, we have $h_1 \neq 1$. All elements $h^{-1} h_1 h$, where $h \in H$, lie in the projection of M_1 onto $\{a_1\}^{G_0}$. If there are only finitely many such elements, then among them there is an element commuting with an infinite set of elements of H. But this is impossible by Lemma 3, and therefore the projection of M_1 onto $\{a_1\}^{G_0}$ is an infinite group. It is proved analogously that if the projection of M_i onto $\{a_i\}^{G_{i-1}}$ is infinite, then the projection of M_{i+1} onto $\{a_{i+1}\}^{G_i}$ is also infinite. Hence it follows that the projection of M_n onto $\{a_n\}^{G_{n-1}}$ is an infinite group.

The nilpotent subgroup F_1, as was shown, intersects nontrivially the normal divisor $H_1 = \{a_{n+1}\}^{G_n}$. Then the center of the group F_1 also intersects H_1 nontrivially. Let $f \in Z(F_1) \cap H_1$, $f \neq 1$. The element f commutes with $M_n \subset F_1$. In this case f also commutes with the projection of M_n onto $\{a_n\}^{G_{n-1}}$. But, since this projection is infinite, this is impossible by Lemma 3. The contradiction obtained proves the theorem.

Consequence. In the sequence

$$\mathfrak{N} \subset R\mathfrak{N} \subset R^2\mathfrak{N} \subset \ldots \subset R^m\mathfrak{N} \subset R^{m+1}\mathfrak{N} \subset \ldots \subset R^\omega\mathfrak{N} \subset R^{\omega+1}\mathfrak{N}$$

all inclusions are proper.

Indeed, the discrete direct product of the groups constructed in Theorem 1 over all n gives a Baer group not lying in $R^\omega\mathfrak{N}$. Therefore neither the equality $R^\omega\mathfrak{N} = R^{\omega+1}\mathfrak{N}$, nor $R^m\mathfrak{N} = R^{m+1}\mathfrak{N}$ for any natural m, can hold.

On the other hand, it is known (^{3}) that for any class of groups $\mathfrak{X}$

$$R^{\omega+1}\mathfrak{X} = R^{\omega+2}\mathfrak{X} = \overline{R}\mathfrak{X}$$

and, in particular, $R^{\omega+1}\mathfrak{N} = \overline{R}\mathfrak{N}$.

2. The operator $\overline{K}$ was first considered by P. G. Kontorovich, who, in particular, considered the class of groups $\overline{K}\mathfrak{A}$, which he called groups with a category (^{4}). The following theorem shows that in a number of cases the operator $\overline{K}$ coincides in its action with the operator of radical closure $\overline{R}$.

Theorem 2. *Let $\mathfrak{X}$ be a class of groups satisfying the following requirements:*

1) *all groups of the class $\mathfrak{X}$ are nonzero;*
2) *in any group the product of two normal divisors from the class $\mathfrak{X}$ belongs to the class $\overline{F}\mathfrak{X}$;*
3) the class $\mathfrak{X}$ is closed with respect to homomorphisms and the taking of subgroups. Then the equality $F\mathfrak{X} = \bar{K}\mathfrak{X} = \bar{R}\mathfrak{X}$ holds.

Proof. For any class of groups $\mathfrak{X}$ there are inclusions $F\mathfrak{X} \subset K\mathfrak{X} \subset R\mathfrak{X}$, whence it follows that $F\mathfrak{X} \subset \bar{K}\mathfrak{X} \subset \bar{R}\mathfrak{X}$. The class $\bar{R}\mathfrak{X}$ is the minimal radical class over $\mathfrak{X}$. We shall prove that if $\mathfrak{X}$ satisfies the conditions of the theorem, then $\bar{F}\mathfrak{X}$ is a radical class, whence the inclusion $\bar{R}\mathfrak{X} \subset \bar{F}\mathfrak{X}$, proving the theorem, will follow.

Since in any group the union of an increasing sequence of normal divisors from $\bar{F}\mathfrak{X}$, obviously, also belongs to $\bar{F}\mathfrak{X}$, it is enough to prove the following lemma.

Lemma 4. If a class of groups $\mathfrak{X}$ satisfies the conditions of the theorem, then in any group the product of two normal divisors from $\bar{F}\mathfrak{X}$ also belongs to the class $\bar{F}\mathfrak{X}$.

Proof. It is enough to show that if some group G is generated by its normal divisors A_1 and A_2, belonging to $\bar{F}\mathfrak{X}$, then $G \in \bar{F}\mathfrak{X}$. For some ordinal number α, $A_1 \in F^{\alpha}\mathfrak{X}$ and $A_2 \in F^{\alpha}\mathfrak{X}$. If $\alpha = 0$, then $G \in F\mathfrak{X}$ by condition 2) of the theorem. Suppose that $\alpha > 0$ and that for all β less than α it has already been proved that in any group the product of two normal divisors from $F^{\beta}\mathfrak{X}$ belongs to the class $\bar{F}\mathfrak{X}$.

We note that, according to conditions 2) and 3) of the theorem, the class of groups $\bar{F}\mathfrak{X}$ is closed with respect to homomorphisms, the taking of subgroups, and finite direct products.

Consider in the group G all possible normal series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots,$$

where G_n is a normal divisor in G_{n-1}, generated by some finite set of elements M_{n-1}. For the group G to belong to the class $\bar{F}\mathfrak{X}$ it is necessary that in each such series, for some natural n, $G_n \in \mathfrak{X}$, and it is sufficient that $G_n \in \bar{F}\mathfrak{X}$.

Writing each $g \in M_n$ in the form $g = a_1 a_2$, $a_1 \in A_1$, $a_2 \in A_2$, we construct finite sets $M_n' \subset A_1$ and $M_n'' \subset A_2$, $M_n \subset M_n' M_n''$, and let

$$G = H_1 \supset H_2 \supset \cdots \supset H_n \supset H_{n+1} \supset \cdots$$

be such a normal series in G that H_n is a normal divisor in H_{n-1}, generated by the set $M_n' \cup M_n''$ and by the subgroup $A_1 \cap A_2 = A$. Since the group G/A is the direct product of its normal divisors A_1/A and A_2/A, by the remark made above it belongs to $\bar{F}\mathfrak{X}$. Consequently, there exists such an n that $H_n/A \in \mathfrak{X}$. We shall prove that then $G_{n+1} \in \bar{F}\mathfrak{X}$, and thereby it will be shown that $G \in \bar{F}\mathfrak{X}$. Consider the normal divisors S, S_1, and S_2 of the group H_n, generated respectively by the sets $M_n' \cup M_n''$, M_n', and M_n''. Since $G_n \subset H_n$, we have $G_{n+1} \subset S$. Let X be a finite set of elements whose image in G/A generates the subgroup H_n/A. Denote by T the subgroup of the group G generated by the set $X \cup M_n' \cup M_n''$, by T_1 the normal divisor of the group T generated by the set M_n', and by T_2 the normal divisor of T generated by M_n''. In view of condition 1) of the theorem, all groups of the class $F\mathfrak{X}$ are locally Noetherian. Therefore the group G, which is generated by its normal divisors A_1 and A_2 belonging to $\bar{F}\mathfrak{X}$, is also locally Noetherian. Consequently, the group T is Noetherian, and its subgroups T_1 and T_2 have finite systems of generators Y_1 and Y_2. It is clear that $Y_1 \subset A_1$ and $Y_2 \subset A_2$. Let B_1 and B_2 be the normal divisors in A_1 and A_2, respectively, generated by the sets Y_1 and Y_2. Since Y_1 and Y_2 are finite sets, $B_1 \in F^{\beta}\mathfrak{X}$ and $B_2 \in F^{\beta}\mathfrak{X}$ for some $\beta < \alpha$.

Since T_1 and A_1 are invariant with respect to X, the subgroup B_1 is also invariant with respect to X; and since H_n is contained in the subgroup generated by the set X and A_1, B_1 is invariant with respect to H_n. Similarly, the subgroup B_2 is invariant with respect to H_n. Moreover, we have $M'_n \subset T_1 \subset B_1$, $M''_n \subset T_2 \subset B_2$, whence $S_1 \subset B_1$ and $S_2 \subset B_2$. Consequently, $S_1 \in F^{\beta}\mathfrak{X}$ and $S_2 \in F^{\beta}\mathfrak{X}$. Therefore, by the induction hypothesis,

$$S = S_1 S_2 \in \bar{F}\mathfrak{X}.$$

Then also $G_{n+1} \in \bar{F}\mathfrak{X}$. The lemma, and together with it the theorem, are proved.

Corollary 1. Every Baer group is a group with categories.

Indeed, the class $\mathfrak{A}_f$ satisfies the condition of Theorem 2 and, consequently,

$$K\mathfrak{A} = K\mathfrak{A}_f = R\mathfrak{A}_f = \mathfrak{B}.$$

Corollary 2. *Every periodic Baer group is contained in the class $F\mathfrak{F}$ of generalized locally normal groups.*

The class of groups $F\mathfrak{F}$ was introduced in the paper (5).

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Note: Figure translations are in progress. See original paper for figures.