ON ONE SOLUTION OF THE ONE-DIMENSIONAL DIFFUSION EQUATION IN A BOUNDED DOMAIN

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Abstract

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ON ONE SOLUTION OF THE ONE-DIMENSIONAL DIFFUSION EQUATION IN A BOUNDED DOMAIN

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A qualitative study of self-oscillations arising under certain conditions in oil production or in some electrolytic systems was given in papers ($^{1-3}$). In papers ($^{3-10}$), periodic solutions of the one-dimensional diffusion equation in a semi-infinite domain of variation of the spatial variable x, connected with this problem, were obtained. Below a periodic solution is found for a certain diffusion problem in a bounded domain.

As is known (^{11}), the diffusion equation

$$\partial c/\partial t = D\,\partial^2 c/\partial x^2 \tag{1}$$

has periodic solutions of the form

$$Ae^{\pm\sqrt{\omega/2D}\,x}\cos\left(\mp\sqrt{\omega/2D}\,x+\omega t\right), \qquad Ae^{\pm\sqrt{\omega/2D}\,x}\sin\left(\mp\sqrt{\omega/2D}\,x+\omega t\right). \tag{2}$$

For the semi-infinite interval $0 < x < \infty$, a periodic solution of equation (1) was obtained in paper (^{4}).

Let us now consider the following problem: to find a periodic solution of equation (1) for the finite interval $0 < x < l$, satisfying the boundary condition

$$c(l,t) = c^0. \tag{3}$$

We shall seek the function $c(x,t)$ in the form

$$c(x,t) = c_0 + (c^0 - c_0)x/l + u(z,\tau), \tag{4}$$

where $u(z,\tau)$ is a periodic solution of the equation

$$\partial u/\partial\tau = \partial^2 u/\partial z^2 \qquad (\tau = Dt/l^2,\ z = x/l - 1) \tag{5}$$

with period T, satisfying the conditions

$$u(0,\tau) = 0, \qquad \int_0^T u(z,\tau)\,d\tau = 0. \tag{6}$$

Using formulas (2), (5), and (6), it is easy to verify that the desired solution can be represented in the form

$$u(z,\tau) = \sum_{k=1}^{\infty} u_k(z,\tau), \qquad u_k(z,\tau) = A_k\{e^{\rho_k z}\cos(\omega_k\tau + \rho_k z) - \tag{7}$$

$$-e^{-\rho_k z}\cos(\omega_k\tau - \rho_k z)\} + B_k\{e^{\rho_k z}\sin(\omega_k\tau + \rho_k z) - e^{-\rho_k z}\sin(\omega_k\tau - \rho_k z)\},$$

where

$$\rho_k = \sqrt{\pi k/T}, \qquad \omega_k = 2\pi k/T, \tag{8}$$

and A_k and B_k denote constants which must be determined from the boundary condition.

We shall now seek a periodic solution of equation (1) for the finite interval $0 < x < l$ with the boundary condition

$$\partial c(0,t)/\partial x = \Omega[c(0,t)], \tag{9}$$

where Ω is an S-shaped function (^{2}), at $x = 0$, and with condition (3) at $x = l$.

Using formulas (4) and (5), we reduce this problem to finding a solution of the diffusion equation (5) in the form (7), where the constants A_k and B_k are found from the boundary condition

$$\partial u(-1,\tau)/\partial z = F[u(-1,\tau)] \quad (F(u) = l\Omega(c_0 + u) - c^0 + c_0). \tag{10}$$

Introducing the notation

$$\begin{aligned} \alpha_k(z) &= 2(A_k\cos\rho_k z\ \mathrm{sh}\,\rho_k z + B_k\sin\rho_k z\ \mathrm{ch}\,\rho_k z), \\ \beta_k(z) &= 2(-A_k\sin\rho_k z\ \mathrm{ch}\,\rho_k z + B_k\cos\rho_k z\ \mathrm{sh}\,\rho_k z), \end{aligned} \tag{11}$$

we write equation (10) in the following form:

$$\sum_{k=1}^{\infty} [\alpha'_k(1)\cos\omega_k\tau + \beta'_k(1)\sin\omega_k\tau] =$$

$$= F\left\{-\sum_{k=1}^{\infty} [\alpha_k(1)\cos\omega_k\tau + \beta_k(1)\sin\omega_k\tau]\right\}. \tag{12}$$

Equation (12), taking into account the notation (11), gives an infinite system of nonlinear equations for finding the constants A_k and B_k $(k = 1, 2, ...)$.

If $F[u(-1,\tau)] = \chi(\tau)$, where $\chi(\tau)$ is a known periodic function of τ with period T, satisfying, by virtue of (6), the condition

$$\int_0^T \chi(\tau)\,d\tau = 0, \tag{13}$$

then, expanding $\chi(\tau)$ in a Fourier series, one can find the coefficients A_k and B_k from (12) and (11).

Let us consider in more detail the case where $F(u)$ is a piecewise-constant function

$$F(u) = Q_1 \quad \text{for } 0 < \tau < T_1; \qquad F(u) = Q_2 \quad \text{for } T_1 < \tau < T. \tag{14}$$

It follows from (6) that in this case the constants Q_1, Q_2, T_1, and T must be related by

$$Q_1T_1 + Q_2(T - T_1) = 0. \tag{15}$$

Formulas (12), (11), and (13) give the following expressions for the coefficients A_k and B_k:

$$A_k = \frac{(Q_1 - Q_2)}{\pi k\rho_k}\sin\frac{\pi kT_1}{T}\left[(\mu_k - \nu_k)\cos\frac{\pi kT_1}{T} - (\mu_k + \nu_k)\sin\frac{\pi kT_1}{T}\right],$$

$$B_k = \frac{(Q_1 - Q_2)}{\pi k\rho_k}\sin\frac{\pi kT_1}{T}\left[(\mu_k + \nu_k)\cos\frac{\pi kT_1}{T} + (\mu_k - \nu_k)\sin\frac{\pi kT_1}{T}\right], \tag{16}$$

$$\mu_k = \operatorname{ch}\rho_k\cos\rho_k/(\operatorname{ch}2\rho_k + \cos 2\rho_k), \qquad \nu_k = \operatorname{ch}\rho_k\sin\rho_k/(\operatorname{ch}2\rho_k + \cos 2\rho_k).$$

We now give the solution of the desired problem in another form. Suppose first that the boundary condition at $z = -1$ has the form

$$\partial u(-1,\tau)/\partial z = \chi(\tau), \tag{17}$$

where the periodic function $\chi(\tau)$, for which equality (13) is satisfied, is known. Consider equation (1). The source function $G(x,t,\xi,\tau)$ for the interval $(0 < x < l)$, satisfying the conditions $\partial G(0,t,\xi,\tau)/\partial x = G(l,t,\xi,\tau) = 0$, has the form (^{11,3}):

$$G(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi D(t-\tau)}} \sum_{k=-\infty}^{\infty} (-1)^k \times$$

$$\times \left[\exp\left(-\frac{(x-\xi-2kl)^2}{4D(t-\tau)}\right) + \exp\left(-\frac{(x+\xi-2kl)^2}{4D(t-\tau)}\right)\right].$$

It can also be represented in the following way:

$$G(x,t,\xi,\tau) = \frac{2}{l}\sum_{k=0}^{\infty} \exp\left[-\frac{(2k+1)^2\pi D}{4l^2}(t-\tau)\right] \times$$

$$\times \cos\frac{(2k+1)\pi\xi}{2l} \cos\frac{(2k+1)\pi x}{2l}.$$

Using the source function, we write the solution of equation (5) for $x = 0$ with boundary conditions (6) and (17)

$$u(0,\tau) = -\int_{-\infty}^{\tau} K(\tau-\sigma)\chi(\sigma)\,d\sigma \quad \left(K(\sigma) = \left[1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\sigma}\right]/\sqrt{\pi\sigma}\right). \tag{18}$$

Let now, analogously to (6),

$$\chi(\tau) = \psi_i(\tau) \quad \text{for } \alpha_i + kT < \tau < \beta_i + kT$$

$$(i = 1,2;\ k = 0, \pm 1, \pm 2, \ldots) \tag{19}$$

$$\alpha_1 = pT/2, \quad \alpha_2 = \beta_1 = T - pT/2, \quad \beta_2 = T + pT/2, \quad 0 < p < 1.$$

It is easy to see that

$$u_i(0,\tau) = -\int_{\alpha_i}^{\tau} K(\tau-\sigma)\psi_i(\sigma)\,d\sigma + S_i(\tau) \quad (\alpha_i \le \tau \le \beta_i), \tag{20}$$

$$S_i(\tau) = -\sum_{k=0}^{-\infty} \left\{ \int_{\beta_i - T + kT}^{\alpha_i + kT} K(\tau - \sigma)\varphi_i(\sigma)\, d\sigma + \int_{\alpha_i - T + kT}^{\beta_i - T + kT} K(\tau - \sigma)\psi_i(\sigma)\, d\sigma \right\}$$

$$(\varphi_1 = \psi_2,\ \varphi_2 = \psi_1).$$

The uniform convergence of the series $S_i(\tau)$ follows from the fulfillment of condition (13) and from the fact that $K(\sigma)$ is a monotonically decreasing function of its argument.

Consider again the case of a piecewise-constant function $F(u)$ of the form (14). Using the following relation from (18),

$$f(a,b) = \int_a^b K(y)\, dy = \frac{2}{\sqrt{\pi}} \left\{ \sqrt{b} - \sqrt{a} + 2\sum_{n=1}^{\infty} (-1)^n \left[n\sqrt{\pi} \left(\Phi\left(\frac{n}{\sqrt{b}}\right) - \Phi\left(\frac{n}{\sqrt{a}}\right) \right) + \sqrt{b} e^{-n^2/b} - \sqrt{a} e^{-n^2/a} \right] \right\}$$

$$\left(\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2}\, dy \right), \tag{21}$$

we write the solution $u(0,\tau)$ in the following way:

$$u_i(0,\tau) = Q_i f(\tau - \alpha_i, 0) + S_i(\tau) \quad (\alpha_i \le \tau \le \beta_i),$$

$$S_i(\tau) = \sum_{k=0}^{-\infty} \{ q_i f(\tau - \beta_i + T - kT,\ \tau - \alpha_i - kT) + Q_i f(\tau - \alpha_i + T - kT,$$

$$\tau - \beta_i + T - kT) \} \quad (q_1 = Q_2,\ q_2 = Q_1). \tag{22}$$

Using condition (13), it is easy to see that $du_1(0,\tau)/d\tau > 0$, $du_2(0,\tau)/d\tau < 0$.

Let us return to the more general case, when the periodic function $\chi(\tau)$, defined by formula (19), is given. From formulas (20) we find

$$\frac{du_i(0,\tau)}{d\tau} = -\frac{1}{\sqrt{\pi}} \left\{ \frac{\psi_i(\alpha_i)}{\sqrt{\tau - \alpha_i}} + \int_{\alpha_i}^{\tau} \frac{\psi_i'(\sigma)\, d\sigma}{\sqrt{\tau - \sigma}} + \right.$$

$$\left. + \int_{\alpha_i}^{\tau} \frac{\psi_i(\sigma)}{(\tau - \sigma)^{5/2}} \sum_{n=1}^{\infty} \left[(-1)^n e^{-n^2/(\tau - \sigma)} (2n^2 - \tau + \sigma) \right] d\sigma \right\} + \frac{dS_i}{d\tau},$$

$$\frac{dS_i}{d\tau} = -\sum_{k=0}^{-\infty} \left\{ \int_{\beta_i - T + kT}^{\alpha_i + kT} K'(\tau-\sigma)\varphi_i(\sigma)\,d\sigma + \int_{\alpha_i - T + kT}^{\beta_i - T + kT} K'(\tau-\sigma)\psi_i(\sigma)\,d\sigma \right\}$$

$$(\alpha_i < \tau < \beta_i). \tag{23}$$

It is easy to see that, by virtue of (13), $dS_1/d\tau > 0$. It can be shown that, if $\psi_1(\tau) < 0,\ \psi_2(\tau) > 0,\ (1-p)T < 2,\ pT < 2$, then, in order for the inequalities $du_1(0,\tau)/d\tau > 0\ (\alpha_1 < \tau < \beta_1)$, $du_2(0,\tau)/d\tau < 0\ (\alpha_2 < \tau < \beta_2)$ to hold, it is sufficient that the relations $(i = 1, 2)$ hold.

$$(-1)^i \left[\frac{\psi_i(\alpha_i)}{\sqrt{\tau - \alpha_i}} + \int_{\alpha_i}^{\tau} \frac{\psi_i'(\sigma)\,d\sigma}{\sqrt{\tau-\sigma}} - \int_{\alpha_i}^{\tau} \frac{e^{-1/(\tau-\sigma)}(2-\tau+\sigma)\psi_i(\sigma)\,d\sigma}{(\tau-\sigma)^{5/2}} \right] > 0. \tag{24}$$

The function F entering condition (10) may then be regarded as having the form

$$F(u, \partial u/\partial\tau) = \begin{cases} F_1(u) & \text{for } \partial u/\partial\tau > 0, \quad F_1 = \psi_1(\tau),\ u = u_1(0,\tau), \\ F_2(u) & \text{for } \partial u/\partial\tau < 0, \quad F_2 = \psi_2(\tau),\ u = u_2(0,\tau). \end{cases}$$

As an example, let us consider the case in which the functions $\psi_i(\tau)$ in formula (19) are prescribed by the dependences

$$\psi_i(\tau) = e_i + d_i(\beta_i - \tau)^{q_i} + f_i(\tau - \alpha_i)^{1/2}; \qquad e_1 < 0,\ d_1 < 0,\ f_1 > 0, \tag{25}$$

$$\frac{1}{2} < q_i < 1; \quad e_2 > 0,\ d_2 > 0,\ f_2 < 0.$$

A periodic solution of equation (1) with conditions (3) and (17), where $\chi(\tau)$ is given by relations (19) and (25), in the semi-infinite domain $0 < x < \infty$ was found in paper (^{6}). The solution of the corresponding problem in the bounded domain $0 < z < 1$ is given by formulas (20), (19), and (25). Moreover, if we introduce the notation

$$I_1(x,q) = \int_x^{\infty} e^{-z} z^{1/2-q}\,dz, \qquad I_2(x,q) = \int_x^{\infty} e^{-z} z^{q-1/2}\,dz,$$

$$I_3(x) = \int_x^{\infty} e^{-z}\sqrt{\frac{z}{x} - 1}\,dz, \qquad I_4(x) = \int_x^{\infty} \frac{e^{-z}}{z}\sqrt{\frac{z}{x} - 1}\,dz, \tag{26}$$

then inequalities (24), sufficient for having $du_1(0,\tau)/d\tau > 0$, $du_2(0,\tau)/d\tau < 0$, take the form $(p_1 = 1-p,\ p_2 = p)$

$$(-1)^i \left[e_i\left(1-2e^{-1/p_{iT}}\right) - d_i\sqrt{p_{iT}}\left\{\frac{(p_{iT})^{q_i-1/2}}{2q_i-1} + 2I_1\left(\frac{1}{p_{iT}}, q_i\right) - \right.\right.$$

$$\left.\left. -I_2\left(\frac{1}{p_{iT}}, q_i\right)\right\} + f_i\sqrt{p_{iT}}\left\{\frac{\pi}{2} - 2I_3\left(\frac{1}{p_{iT}}\right) + I_4\left(\frac{1}{p_{iT}}\right)\right\}\right] > 0. \qquad (27)$$

From (13) and (25) there also follows a relation among the constants e_i, d_i, f_i, q_i, p and T, coinciding with equality (4.4) of article (⁶):

$$e_2 pT + \frac{d_2}{q_2+1}(pT)^{q_2+1} + \frac{2}{3}f_2(pT)^{3/2} + e_1(1-p)T + \frac{d_1}{q_1+1}[(1-p)T]^{q_1+1} + \frac{2}{3}f_1[(1-p)T]^{3/2} = 0. \qquad (28)$$

From (26) it follows that $\lim_{x\to\infty} I_j(x,q_i) = 0$ $(j = 1,2,3,4)$ and $\lim_{x\to\infty}[(2q_i - 1)x^{q_i-1/2}I_i(x,q_i)] = 0$ $(i = 1,2)$. Therefore, in the limit as $T \to 0$, inequalities (27) pass into inequalities (4.3) of article (⁶).

Thus, if there exists a solution of the problem in the semi-infinite domain, then, at least for sufficiently small values of the period T, there also exists a solution of the corresponding problem in the finite domain. (Indeed, $T \to 0$ $(l \to \infty)$, since from formula (5) it follows that the period T is related to the actual period T_* of the self-oscillation by the dependence $T = DT_*/l^2$.)

Putting in formulas (20) $\psi_1(\sigma) = F_1[u_1(0,\sigma)]$, $\psi_2(\sigma) = F_2[u_2(0,\sigma)]$, we obtain a nonlinear integral equation for finding the function $u(0,\sigma)$ $(\alpha_1 \le \sigma \le \beta_2)$.

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Note: Figure translations are in progress. See original paper for figures.

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