ON PERIPHERALLY BICOMPACT TREE-LIKE SPACES

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Abstract

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V. V. PROIZVOLOV

ON PERIPHERALLY BICOMPACT TREE-LIKE SPACES

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G. L. Gurin proved ([1]) that a peripherally bicompact tree-like* space has a base of connected open sets with finite boundaries and that it is linearly connected, i.e., every pair of points in it is joined by an ordered continuum. These facts will be used as starting points in the present paper.

Theorem 1. *The density** of a peripherally bicompact tree-like space coincides with its weight. In particular, a separable peripherally bicompact space has countable weight.*

Lemma 1. *A pair of points $a, b \in X$ of a tree-like peripherally bicompact space can be joined in it by a unique ordered continuum.*

Proof. Suppose the contrary: L_1 and L_2 are distinct ordered continua joining the points a and b. There exist points $x_1 \in L_1$ and $x_2 \in L_2$ such that $x_1 \notin L_2$ and $x_2 \notin L_1$. We note that $L_1 \setminus x_1 = C_{11} \cup C_{12}$, where C_{11} and C_{12} are open-and-closed in $L_1 \setminus x_1$, and $a \in C_{11}$, $b \in C_{12}$, $C_{11} \cap C_{12} = \Lambda$. Correspondingly, $L_2 \setminus x_2 = C_{21} \cup C_{22}$, $a \in C_{12}$, $b \in C_{22}$, $C_{21} \cap C_{22} = \Lambda$. In view of the tree-likeness of X, there exists a point $c \in X$ separating X between x_1 and x_2, and hence also separating the connected set $L = L_1 \cup L_2$ between the same points. Thus, $L \setminus c = B_1 \cup B_2$, where $x_1 \in B_1$, $x_2 \in B_2$, $B_1 \cap B_2 = \Lambda$. But either $c \in C_{11} \cup C_{21}$, or $c \in C_{12} \cup C_{22}$, each of which is connected. Let, for definiteness, $c \in C_{11} \cup C_{21}$; then c does not separate the connected set $[C_{12} \cup C_{22}]$. The latter contradicts the fact that c separates L between x_1 and x_2, since $x_1, x_2 \in [C_{12} \cup C_{22}]$. The lemma is proved.

Proof of the theorem. Let X be a separable peripherally bicompact tree-like space, and let $A = \{a_i\}$ be a countable dense subset of it. We shall prove that the weight of X is countable. The general case is proved analogously.

Join pairwise all points of A by ordered continua and enumerate them in an arbitrary order $L_1, L_2, \dots, L_k, \dots$. All possible finite unions of these continua shall also be enumerated in an arbitrary way $F_1, F_2, \dots, F_s, \dots$. The locally connected set $X \setminus F_s$ splits into open components, of which there will be no more than countably many in view of the separability of X,

$$X \setminus F_s = \bigcup_{i=1}^{\infty} V_{si}.$$

It is asserted that the countable system of open sets $V = \{V_{si}\}$, over admissible s and i, is a base in X. We shall prove this.

If Ox is an arbitrary neighborhood of a point $x \in X$, then there is a connected neighborhood $O'x$ with finite boundary ([1]), $[O'x] \setminus O'x = (x_1, \dots, x_k)$, and such that $[O'x] \subset Ox$. For all points x_i take connected neighborhoods

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* A connected space is tree-like if it can be separated between any two of its points by some third point.

** Density is the cardinality of the smallest dense subset.

Ox_i such that $x \notin Ox_i$ and such that $Ox_i \cap Ox_j = \Lambda,\ i \neq j$. If $Ox_i \setminus [O'x] \neq \Lambda$, then we choose arbitrarily two points from A: a point $a_{i_1} \in Ox_i \setminus [O'x]$ and a point $a_{i_2} \in Ox_i \cap O'x$. Join the points a_{i_1} and x_i by an ordered continuum L_{1i}, and also the points x_i and a_{i_2} by an ordered continuum L_{2i}. The bicompactum $L_i = L_{1i} \cup L_{2i}$ is, by the ordering lemma, a unique ordered continuum joining the points a_{i_1} and a_{i_2}, $L_i \subset Ox_i$. The set $X \setminus F$, where $F = \bigcup_i L_i$, decomposes into open components that are elements of the system V; the one of them that contains the point x lies entirely in Ox. The theorem is proved.

Theorem 2. *If V is an open F_σ-set with connected closure in a tree-like bicompactum X, then the boundary of V is metrizable.*

Proof. By hypothesis $V = \bigcup_1^\infty F_i$, where F_i is a bicompactum for every i; moreover, we may assume that $F_i \subset F_j$ if $i < j$. For each F_i there exists a neighborhood $OF_i \subset V$ with finite boundary Γ_i, which does not meet the boundary of V. Obviously, the set $\Gamma = \bigcup_1^\infty \Gamma_i$ is countable and $[\Gamma]$ contains the boundary of V.

Note that $[V] = Y \subset X$ and that, since Y is a connected closed subset of X, Y is a tree-like bicompactum. In what follows all operations are performed in Y. Join the points of the set Γ pairwise in Y by ordered continua; we obtain a countable system of ordered continua $\{L_i\}$. Finite unions of elements of this system form a countable system $\{S_i\}$. Finally, introduce the countable system of open subsets of Y, $\{Q_i\}$, where $Q_i = Y \setminus S_i$.

Along with $\{Q_j\}$, consider the countable system of open sets $\{P_i\}$, where $P_i = Y \setminus [OF_i]$. The system $\{M_i\}$ is the countable system of all possible pairwise intersections between elements of the systems $\{Q_i\}$ and $\{P_i\}$. Finally, in view of the local connectedness of the space Y, each element M_i decomposes into open components, $M_i = \bigcup_s M_{is}$. Take those elements of the system $\{M_{is}\}$ that meet the countable set Γ. As a result we obtain a countable system of open sets $\{N_j\}$. The system $\{N_i\}$ is a countable base of the boundary of V.

Indeed, let $x \in [V] \backslash V$ and let Ox be an arbitrary neighborhood of it; choose such a connected neighborhood O_1x of the point x that $[O_1x] \subset Ox$ and the boundary of O_1x is finite. The points $A = (a_1, \dots, a_k)$ are those points of the boundary of O_1x that belong to the boundary of V, and the points $B = (b_1, \dots, b_m)$ are those points of the boundary of O_1x that belong to V. There exists such an OF_i that $B \subset OF_i$, and this means that $B \cap P_i = \Lambda$. For every point $a_i \in A$ take such a connected neighborhood Oa_i that its boundary is finite and such that $[Oa_i] \cap (A \cup B) = a_i$. There exist a point $c_{1i} \in \Gamma$ and a point $c_{2i} \in \Gamma$ such that $c_{1i} \in Oa_i \setminus O_1x$, and the point $c_{2i} \in Oa_i \cap O_1x$. Join the points c_{1i} and a_i by an ordered continuum L_{1i}, and the points c_{2i} and a_i by an ordered continuum L_{2i}. Then $L_i = L_{1i} \cup L_{2i}$ is an ordered continuum, since $L_{1i} \subset Oa_i$, $L_{2i} \subset Oa_i$. There exists such a natural number l that $Y \setminus (L_1 \cup \dots \cup L_k) = Q_l$, where $Q_l \in \{Q_i\}$. It is now clear that that component of the set $P_i \cap Q_l$ which contains the point x is contained in Ox, and it is an element of the system $\{N_i\}$. The theorem is proved.

Theorem 3. *A tree-like peripherally bicompact space X has a tree-like bicompact extension, and a unique one. The weight of this extension coincides with the weight of X.*

Lemma 2. *The intersection of any number of connected closed subsets of a peripherally bicompact tree-like space is connected or empty.*

Proof. Let $\bigcap F_\alpha = F$, where $\{F_\alpha\}$ are connected and closed. Suppose that F is disconnected: $F = F_1 \cup F_2$, $F_1 \cap F_2 = \Lambda$. Take points $x_1 \in$

$\in F_1$ and $x_2 \in F_2$. There exists an ordered continuum L_α joining x_1 and x_2 in F_α. Among the continua $\{L_\alpha\}$ there must be distinct ones, for otherwise x_1 and x_2 would be contained in one component of the set F. But this contradicts Lemma 1.

Lemma 3. *Let X be a tree-like space. Let the sets $A, B \subset X$ be connected, closed, and disjoint. Then there exists a point $c \in X$ which separates A and B.*

Proof. Denote $X_A = X \setminus A$; take that connected component P of the set X_A which contains B, $B \subset P$. The set $[P] \setminus P$ consists of only one point; suppose the contrary: $a, b \in [P] \setminus P$, then there exists a point d separating a and b, $X \setminus d = X_1 \cup X_2$, $X_1 \cap X_2 = \Lambda$, $a \in X_1$, $b \in X_2$. But then P belongs either to X_1 or to X_2; let $P \subset X_1$. We have

$$[P] \subset [X_1] = X_1 \cup d \not\ni b.$$

But the set $[P]$ contains the point b—we have obtained a contradiction. Thus, the set $[P] \setminus P$ is a singleton, $[P] \setminus P = s$.

Proceeding in the same way with $X_B = X \setminus B$, we find a connected component Q of the set X_B containing A, $A \subset Q$, and denote $[Q] \setminus Q = t$, where t is a point. Obviously, $s \in A$, $t \in B$.

Take a point c separating s and t; it separates the sets A and B. Indeed, since P is connected and $[P]$ is also connected and, moreover, the points $s, t \in [P]$, the point c splits the set $[P]$, and consequently the point $c \in P$. In exactly the same way we show that $c \in Q$. Thus the point $c \in P \cap Q$, i.e. $c \notin A \cup B$. The last fact, together with all that precedes it, means that the point c separates A and B. The lemma is proved.

Proof of Theorem 3. Denote by $\hat{X}$ the set of all maximal centered systems of connected closed subsets of a tree-like peripherally bicompact space. A maximal centered system of connected closed subsets will here, for brevity, be called an end. We introduce a topology on the set of ends $\hat{X}$ in the following way. The set of ends containing as an element a closed connected set $A \subset X$ is declared closed in $\hat{X}$ and is denoted by $\hat{A}$. Closed sets of the indicated form constitute, by definition, a pseudobase of closed sets in $\hat{X}$. The system $\{O_A\}$, where O_A is the set of ends having an element in $X \setminus A$, is a pseudobase of open sets in $\hat{X}$.

By Alexander' s lemma ([2]), in order to prove bicompactness of $\hat{X}$ it suffices to prove that every centered system of closed sets from elements of the closed pseudobase of the space $\hat{X}$ has nonempty intersection. The latter follows from the fact that if $\{\hat{A}_\alpha\}$ is a centered system of closed sets in $\hat{X}$, then $\{A_\alpha\}$ is the same kind of system in X, and $\bigcap_\alpha \hat{A}_\alpha$ consists of all those ends which contain the system $\{A_\alpha\}$ as a subsystem, i.e. is nonempty.

The space $\hat{X}$ is tree-like. Let $a = \{A_\alpha\}$ and $b = \{B_\alpha\}$ be two ends in X, i.e. two points in $\hat{X}$. By Lemma 2 there are such A_α and B_α that $A_\alpha \cap B_\alpha = \Lambda$. By Lemma 3 there exists a point $c \in X$ separating the connected A_α and B_α. The end consisting of connected closed sets containing the point c separates the points a and b in $\hat{X}$.

The natural embedding X in $\hat{X}$ is a homeomorphism, which is proved directly using the fact that at all points of X there are bases of connected neighborhoods with finite boundaries.

It can be shown, using Lemma 1, that the remainder $\hat{X} \setminus X$ is punctiform*. Suppose that two tree-like bicompact extensions have been found—

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* A space is punctiform if it contains no connected non-singleton subsets.

extensions $b_1 X$ and $b_2 X$ for X, one must conclude that the natural mapping $\varphi : b_2 X \to b_1 X$ of one onto the other is one-to-one; otherwise it is easy to obtain a contradiction with Lemma 2. Thus, X has a unique tree-like bicompact extension.

The weight of $\widetilde{X}$ is equal to the weight of X, which follows immediately from Theorem 1. Theorem 3 is proved.

In conclusion we formulate an unsolved problem, whose possible solution will clarify the place occupied by tree-like bicompacts in the family of all bicompacts.

Is every tree-like bicompact a continuous image of an ordered continuum?

Department of Mechanics and Mathematics
of Moscow State University
named after M. V. Lomonosov

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References

[1] G. L. Gurin, *Vestn. Moskovsk. Univ.*, No. 1, 9 (1969). [2] J. L. Kelley, *General Topology*, "Nauka," 1968.

Note: Figure translations are in progress. See original paper for figures.