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Abstract

Full Text

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ON A METHOD FOR SOLVING NONLINEAR FUNCTIONAL EQUATIONS

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1°. Let $P(x) = 0$ be a nonlinear functional equation having a unique root x^* in some domain $R = \{\|x - x^*\| \leq \eta\}$, $\eta = \text{const} > 0$, and let the operation $P(x)$ map elements of a normed space X of type B into elements of a normed space Y of the same type. We assume that $P(x)$ is differentiable (in the Fréchet sense, ([1])) $k+1$ times and that the inverse operation $[P'(x)]^{-1}$ exists. We seek a method of successive approximations in the form

$$x_{n+1} = x_n + \sum_{i=1}^{k} C_i(x_n) P^i(x_n) = F(x_n), \tag{1.1}$$

where $C_i(x)$ are certain operations. We shall choose them so that method (1.1) converges to the root of the equation.

Differentiating $F(x)$ with respect to x,

$$F'(x) = I + \sum_{i=1}^{k} \{C_i'(x) P^i(x) + i C_i(x) P^{i-1}(x) P'(x)\}$$

or

$$F'(x) = \sum_{i=0}^{k-1} \{C_i'(x) + (i+1) C_{i+1}(x) P'(x)\} P^i(x) + C_k'(x) P^k(x); \qquad C_0(x) = x.$$

We choose $C_i(x)$ so that

$$C_0(x) \equiv x, \qquad C_i'(x) + (i+1) C_{i+1}(x) P'(x) = 0, \quad i = 1, \ldots, k-1. \tag{1.2}$$

Solving system (1.2) with respect to $C_i(x)$, we obtain

$$C_i(x) = -C_{i-1}'(x) / i P'(x), \qquad C_0(x) \equiv x, \quad i = 1, \ldots, k. \tag{1.3}$$

Now method (1.2) can be written in the form

$$x_{n+1} = x_n + \sum_{i=1}^{k} C_i(x_n) P^i(x_n), \qquad C_i(x_n) = -\frac{C'_{i-1}(x_n)}{iP'(x_n)}, \qquad C_0(x) \equiv x. \tag{1.4}$$

Theorem. The method of successive approximations (1.4), for an initial approximation x_0 sufficiently close to x^*, converges with a rate characterized by the inequality

$$\|x_{n+1} - x_n\| \leq q^{[(k+1)^n - 1]/k} \|x_0 - x^*\|, \qquad 0 < q < 1. \tag{1.5}$$

Proof. By virtue of (1.3) we have

$$F'(x) = C'_k(x) P^k(x),$$

whence it follows that

$$F'(x^*) = F''(x^*) = ... = F^{(k)}(x^*) = 0, \qquad F^{(k+1)}(x^*) = k! C'_k(x^*) {P'}^k(x^*). \tag{1.6}$$

Without loss of generality in the proof we assume that $F^{(k+1)}(x^*) \neq 0$. Using an analogue of Taylor' s formula for the case of nonlinear functional equations ([1]), we obtain

$$\left\| F(x_n) - F(x^*) - (x_n - x^*) F'(x^*) - ... - \frac{(x_n - x^*)^k}{k!} F^{(k)}(x^*) \right\| \leq$$

$$\leq \frac{\|x_n - x^*\|^{k+1}}{(k+1)!} \sup_{\substack{\tilde{x} = x_n + \theta_n (x_n - x^*) \\ 0 \leq \theta_n \leq 1}} \left\| F^{(k+1)}(\tilde{x}) \right\|.$$

Hence, in view of (1.6), we have

$$\|x_{n+1} - x^*\| \leq \frac{M_{k+1}}{(k+1)!} \|x_n - x^*\|^{k+1} \quad \left(M_{k+1} = \sup_{\substack{x \in \rho, \ \|x - x^*\| \leq \eta_0 \\ \eta_0 = \|x_0 - x^*\| < \eta}} \left\| F^{(k+1)}(x) \right\| \right),$$

and after obvious transformations, for $\|x_0 - x^*\| < 1$,

$$\frac{M_{k+1}}{(k+1)!}\|x_0 - x^*\| = q < 1,$$

we obtain the validity of (1.5), which indicates a very high rate of convergence of the method under consideration. The theorem is proved.

It follows from what has been obtained that method (1.4) is an iterative method of order $(k+1)$.

2^0. Let us consider some special cases of method (1.4).

1. Setting $k = 1$ in (1.4), we obtain the iterative method of second order

$$x_{n+1} = x_n - P(x_n)/P'(x_n),$$

which is widely known as Newton' s method for solving functional equations.

2. Setting $k = 2$ in method (1.4), we obtain the iterative method of third order

$$x_{n+1} = x_n - P(x_n)/P'(x_n) - P''(x_n)P^2(x_n)/2{P'}^3(x_n). \tag{2.1}$$

3^0. Let us consider some applications of method (2.1).

a) Let $P(x) = 0$ be an algebraic or transcendental equation. Then the successive approximations will be determined by formula (2.1), where $P(x)$ is some function of a real or complex variable.

b) We now apply method (2.1) to the solution of m algebraic (transcendental) equations with m unknowns

$$f_k(\xi_1, \dots, \xi_m) = 0, \qquad k = 1, \dots, m. \tag{3.1}$$

We rewrite method (2.1) in the form *

$$P^3_{n'}(x_{n+1} - x_n) = -(P^2_{n'} - \tfrac{1}{2}P''_n P_n)P_n.$$

Taking into account the meaning of the operations $P'(x)$, $P''(x)$ in the case of systems of algebraic (transcendental) equations ([1]), we determine the successive approximations from the system of equations for the corrections

$$\sum_{j=1}^{m} a_{ij}^{(n)}\left(\xi_j^{(n+1)} - \xi_j^{(n)}\right) = -b_i^{(n)}, \qquad i = 1, \dots, m,$$

* Notation: $P_n = P(x_n)$, $P'_n = P'(x_n)$, $P''_n = P''(x_n)$.

$$a_{ij}^{(n)} = \sum_{s,r=1}^{m} \left(\frac{\partial f_i}{\partial \xi_s}\right)_n \left(\frac{\partial f_s}{\partial \xi_r}\right)_n \left(\frac{\partial f_r}{\partial \xi_j}\right)_n,$$

$$b_i^{(n)} = \sum_{\nu,\mu=1}^{m} (f_\nu)_n \left\{ \left(\frac{\partial f_i}{\partial \xi_\mu}\right)_n \left(\frac{\partial f_\mu}{\partial \xi_\nu}\right)_n + \frac{1}{2}(f_\mu)_n \left(\frac{\partial^2 f_i}{\partial \xi_\nu \partial \xi_\mu}\right)_n \right\}.$$

As is evident, when solving system (3.1) by method (2.1), as in the case of Newton' s method, at each iteration step it is necessary to solve a system of linear algebraic equations of order m; however, the convergence rate of method (2.1) is substantially higher than the convergence rate of Newton' s method. Thus, for example, if $q < 10^{-1}$ and $\|x_0 - x^*\| < 1$, then in the case $k = 1$ (Newton' s method) we have

$$\|x_1 - x^*\| < 10^{-1}, \qquad \|x_2 - x^*\| < 10^{-3},$$

$$\|x_3 - x^*\| < 10^{-7}, \qquad \|x_4 - x^*\| < 10^{-15},$$

whereas for $k = 2$ (method 2.1),

$$\|x_1 - x^*\| < 10^{-1}, \qquad \|x_2 - x^*\| < 10^{-4},$$

$$\|x_3 - x^*\| < 10^{-13}, \qquad \|x_4 - x^*\| < 10^{-40}.$$

As is evident, the number of correct decimal digits in the approximations obtained by method (2.1), in comparison with the approximations obtained by Newton' s method, grows very rapidly.

Example. Let us refine an approximate solution $\xi_1^{(0)} = 0.28$, $\xi_2^{(0)} = 0.77$ of the system $f_1(\xi_1, \xi_2) = \xi_1^2 + \xi_2^3 + \xi_1 - 0.902 = 0$, $f_2(\xi_1, \xi_2) = \xi_1^3 + \xi_2^2 + \xi_2 - 1.467 = 0$. The first approximation by method (2.1) gives $\xi_1^{(1)} = 0.3012$, $\xi_2^{(1)} = 0.8004$. The true value of the root is $\xi_1^* = 0.3$, $\xi_2^* = 0.8$.

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REFERENCES

1. L. V. Kantorovich, *UMN*, 3, no. 6 (1948).

Note: Figure translations are in progress. See original paper for figures.