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Abstract

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MATHEMATICS

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ON CONNECTIONS BETWEEN THE STRUCTURE OF CERTAIN NILPOTENT ASSOCIATIVE ALGEBRAS AND THE STRUCTURE OF THE LATTICE OF THEIR IDEALS

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Consider the class $\mathfrak{N}$ of commutative nilpotent (associative) algebras over the field of complex numbers. Along with each algebra in $\mathfrak{N}$, consider its associated lattice, i.e. its complete lattice of ideals with the natural operations (see, for example, ([1])). The algebras in the class $\mathfrak{N}$ are, in general, not uniquely determined by their associated lattices: thus, for example, it can be shown that any nonisomorphic six-dimensional algebras from the series constructed by D. A. Suprunenko ([2]) have isomorphic associated lattices.

Nevertheless, a number of important properties of a given algebra from the class can be expressed in the language of its associated lattice. Several results of this kind are given in the first part of the present paper. In particular, it is shown that from the position of certain ideals in the associated lattice one can judge into what factors their elements decompose. As a consequence, in the second part of the paper it is obtained that algebras which, in a certain sense, are sufficiently rich in such ideals are determined by their associated lattices completely, i.e. uniquely up to isomorphism. Finally, it is shown that for finite-dimensional algebras this "richness" of the associated lattice is guaranteed by certain relations between the dimension of the algebra and the number of its generators.

1. Let S be some (for the time being arbitrary) complete lattice with operations $\vee, \wedge$ and order relation $<$, as well as with unit 1 and zero 0. If $I, J \in S$, $I < J$ are such that there is no $K \in S$, $I < K < J$, we shall say that the element I immediately precedes the element J and use the notation $I \prec J$.

For $I, J \in S$, $J \leq I$, put

$$S_J^I = \{K \in S : \ J \leq K \leq I\};$$

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obviously, S_J^I is a (complete) sublattice in S with unit I and zero J. For brevity we shall denote S_0^I by S^I, and S_J^1 by S_J. The lattice S will be called elementary if its unit

$$1 = \vee\{I:\ 0 \prec I\}.$$

An important example is the lattice of all subspaces of a complex space of dimension α (α is a cardinal number), denoted by $\Sigma(\alpha)$. For an element $I \in S$, by M^I (respectively, M_I) we shall denote the set of those $J \in S$ for which the lattice S_J^I (respectively, S_I^J) is elementary.

Let

$$0 = I_0 < I_1 < \cdots < I_m = 1$$

be an ordered chain of elements of the lattice S. We shall call it an upper normal series if, for every $k:\ 1 \le k \le m$, I_{k-1} is the minimal element of the set M^{I_k}; and a lower normal series if, for every $k:\ 0 \le k \le m-1$, I_{k+1} is the maximal element of the set M_{I_k}. Finally, an arbitrary set $M \subseteq S$ will be called independent if none of its elements precedes the join of the other elements of this set.

2. Now let $R \in \mathfrak{R}$ and S be its associated lattice. The following assertions are easily proved.

Lemma 1. *For every $I \in S$, among the linearly ordered subsets of the sublattice S^I there exists a set of greatest cardinality, and this cardinality is equal to the dimension of the ideal I.*

In particular, $\dim I = 1$ then and only then, when $0 \prec I$.

Lemma 2. The annihilator A of the algebra R is the union (in the lattice-theoretic sense) of its one-dimensional ideals.

In particular, the algebra is trivial, i.e. has zero multiplication, then and only then, when its ideal lattice is elementary. In this case the ideal lattice is isomorphic to $\Sigma(a)$ for some a.

Lemma 3. For every $I \in S$, the sublattice S_I is, up to isomorphism, the ideal lattice of the factor-algebra R/I.

Lemma 4. The lattice S has a unique upper normal series

$$1 = R_1 > R_2 > \cdots > R_n > R_{n+1} = 0,$$

where the ideal R_k $(1 \le k \le n+1)$ is the set of homogeneous polynomials of degree k in the elements of the algebra R.

Lemma 5. The lattice S has a unique lower normal series

$$0 = A_0 < A_1 < \cdots < A_{n-1} < A_n = 1,$$

where the ideal A_k is the annihilator of the ideal R_k $(1 \le k \le n)$.

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In particular, the upper and lower normal series contain the same number of terms, equal to the nilpotency index $n+1$ of the algebra R.

Lemma 6. For every $I \in S$, in the set M^I there exists a unique minimal element J, consisting of all possible sums of the form

$$\sum_{i=1}^{m} x_i y_i, \quad x_i \in I;\ y_i \in R$$

(m is any natural number). Moreover, among independent sets of ideals $K \in S_J^I,\ J \prec K$, there exists a set of maximal cardinality, and this cardinality is equal to the cardinality of the set of elements of the algebra that generate the ideal I.

In particular, the ideal I is principal then and only then, when it is immediately preceded by only one element of the lattice S.

3. **Lemma 7.** An element $z \in R$ decomposes into a product of m factors if and only if there exists a chain

$$I_z = I_1 < I_2 < \cdots < I_m,$$

consisting of m principal ideals in R.

4. **Lemma 7.** An element $z \in R$ decomposes into a product of m factors ($n+1$ is the nilpotency index of the algebra R); we shall call it decomposable; for such an element, obviously, $0 \leq I_z$. Let $z \in R$ be a decomposable element. We shall say that a principal ideal I has degree k relative to z if there exists a chain of principal ideals of the form

$$I_z = I_1 < \cdots < I_k = I < \cdots < I_n.$$

To each such ideal we assign the ideal

$$J = I \vee A_{k-1} \quad (1 \leq k \leq n),$$

which we shall call the ideal of degree $k-1$ adjoined to z. The subset of S consisting of all ideals adjoined to z will be denoted by $\Omega(z)$.

Consider the set $\Pi_m^{l_1,\ldots,l_m}$ ($m, l_1, \ldots, l_m$ are natural numbers) of monomials in m symbols $X_1, \ldots, X_m$, having the form

$$X_1^{k_1} \cdots X_m^{k_m},$$

where

$$0 \leq k_1 \leq l_1, \ldots, 0 \leq k_m \leq l_m.$$

Introduce on this set an order relation by putting

$$X_1^{i_1} \cdots X_m^{i_m} \leq X_1^{j_1} \cdots X_m^{j_m}$$

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then and only then, when

$$i_1 \geq j_1, \dots, i_m \geq j_m.$$

A decomposable element $z \in R,\ z = x_1 \cdots x_n$, will be called uniquely decomposable if, for every decomposition $z = y_1 \cdots y_n$, up to the order of the factors,

$$y_k = \lambda_k x_k + u_k,$$

where λ_k is a complex number and $u_k \in A_{n-1}$.

Lemma 8. Let

$$z = \lambda x_1^{l_1} \cdots x_m^{l_m}; \quad l_1 + \dots + l_m = n$$

be uniquely decomposable, and suppose all ideals

$$J^{(k)} = I_{x_k} \vee A_{n-1}, \quad 1 \leq k \leq m,$$

are distinct. Then between the partially ordered sets $\Omega(z)$ and $\Pi_m^{l_1,\dots,l_m}$ there exists an order isomorphism, under which to the ideal $J^{(k)} \in \Omega(z)$ there corresponds the monomial

$$X_k \in \Pi_m^{l_1,\dots,l_m} \quad (1 \leq k \leq m).$$

What is important for the sequel is that, in a certain situation, one can prove the assertion converse to Lemma 8.

Theorem 1. Let $z \in R$ be a decomposable element such that every

an independent set of maximal elements in $\Omega(z)$ consists of no more than two elements. Suppose, further, that for some natural numbers m and $l_1, \dots, l_m$, $l_1 + \dots + l_m = n$, and for some elements $x_1, \dots, x_m \in R$, there is an order isomorphism between the sets $\Pi_m^{l_1,\dots,l_m}$ and $\Omega(z)$ taking the monomial X_k to the ideal $J^{(k)} = I_{x_k} \vee A_{n-1}$ $(1 \leq k \leq m)$. Then z is uniquely decomposable, and for some complex λ,

$$z = \lambda x_1^{l_1} \cdots x_m^{l_m}.$$

4. Beginning with this point, for simplicity of exposition we shall consider only algebras for which $n = 2$, i.e. $R_3 = (0)$. In this case Theorem 1 together with Lemma 8 assumes the most transparent form.

Theorem 1′. For an element $z \in R'$ the following two assertions are equivalent: a) z is uniquely decomposable and has the form $z = \lambda xy;\ x, y \in R$; b) I_z is preceded by exactly two ideals of the form $I_u \vee A$, where $I_z < I_u$, namely by the ideals $I_x \vee A$ and $I_y \vee A$.

In what follows, for brevity, we shall call *lines* those ideals $I \in S_A$ for which $A < I$, and *planes* those ideals $I \in S_A$ for which $A < J < I$ for some $J \in S_A$.

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A certain set Ω of planes in S_A will be called *connected* if for any $I, J \in \Omega$ there exists a finite chain $K_1 = I, K_2, \dots, K_i, \dots, K_n = J$ such that $K_i \wedge K_{i+1} > A;\ 1 \leq i \leq n-1$.

A set Ω of planes in S_A will be called *absorbing* if for every line $I \in S_A$ there is a plane $J \in S_A$, $I < J$, and at least three planes $K_i \in \Omega,\ i = 1, 2, 3$, such that $J \wedge K_i > A;\ i = 1, 2, 3$.

5. We now pass to the conditions under which the algebras under consideration are determined uniquely up to isomorphism by their associated structures.

A collection of two lines $J, K \in S_A$ will be called a *regular pair* if for them the situation described in Theorem 1′ occurs; in other words, if there exists an ideal $I \in S,\ 0 < I$, such that $I < I_1 \leq J,\ I < I_2 \leq J$ for some principal ideals $I_1, I_2 \in S$, and moreover for every principal ideal $I_3 > I$ the ideal $I_3 \vee A$ coincides either with J or with K. A plane $I \in S_A$ will be called *marked* if it contains at least four regular pairs of lines such that all 8 lines are distinct.

Theorem 2. Let an algebra $R \in \mathfrak{R}$ $(n = 2)$ have an associated structure S such that in S_A there exists a connected absorbing set consisting of marked planes. Then every algebra $R' \in \mathfrak{R}$ whose associated structure is isomorphic to S is isomorphic to the algebra R.

In the proof the principal role is played by the following auxiliary assertion, which unites all the technical difficulties. Consider a subspace B in R such that $R = B \oplus A$, its tensor square $B \otimes B$, and the operator $\tau : B \otimes B \to R_2$, defined by the equality $\tau(x \otimes y) = xy;\ x, y \in B$.

Lemma 9. Let $T : B \otimes B \to R_2$ be a linear operator having the following property: for every $x, y \in B$ such that the ideals $I = \{\lambda x\} \oplus A$ and $J = \{\lambda y\} \oplus A$ form a regular pair, the vector $T(x \otimes y)$ is collinear with xy. Then, under the hypotheses of Theorem 2, T is a multiple of the operator τ.

6. Let now R be an algebra with m generators $x_1, \dots, x_m$, and, as above, $R_3 = (0)$. Consider the free algebra P of the same type (i.e. $P \in \mathfrak{R};\ P_3 = (0)$) with m generators $X_1, \dots, X_m$, and an epimorphism of algebras $\pi : P \to R$ such that $\pi(X_i) = x_i,\ 1 \leq i \leq m$. Evidently, $\operatorname{Ker} \pi$ is contained in the subspace $P^2 \subset P$, consisting of homogeneous polynomials of the second degree in $X_1, \dots, X_m$; moreover, π maps isomorphically the subspace $P^1 \subset P$ of linear combinations of the elements $X_1, \dots, X_m$ onto a subspace L in R such that $R = L \oplus R_2$.

Denote by $\omega : \mathfrak{M} \to \Gamma^1_m$ the natural one-to-one correspondence between the set

$$\mathfrak{M} = \{I \in S :\ R_2 < I\}$$

and the set Γ^1_m of all lines in L.

It is easily established.

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Lemma 10. Let in the algebra R one have $A = R_2$. Further, let $I, J \in \mathfrak{R}$, $I \neq J$, and let $x, y \in P^1$ be such that $\pi(x) \in \omega(I)$, $\pi(y) \in \omega(J)$. In order that the ideals I and J form a proper pair, it is necessary and sufficient that, for every $z \in \operatorname{Ker} \pi$, $z \neq 0$, the element $xy + z$ should not be decomposable in the algebra P.

As follows from the theory of (complex) quadratic forms, every element $z \in P^2$ can be written in the form $z = y_1^2 + \dots + y_{r(z)}^2$, where $y_1, \dots, y_{r(z)} \in P^1$ are linearly independent; moreover, the space $H(z)$—the linear span of the vectors $y_1, \dots, y_{r(z)}$—and the number $r(z)$—the rank of the element z—are determined by this element uniquely.

Consider the complex Grassmann manifold Γ_m^2 of two-dimensional subspaces in L. To each element $z \in \operatorname{Ker} \pi$ assign a certain subset $\mathfrak{R}(z) \subseteq \Gamma_m^2$, defined in dependence on the rank of z as follows:

- $\mathfrak{R}(z) = \{E \in \Gamma_m^2 : \ \pi[H(z)] \subset E\}$, if $r(z) = 1$,
- $\mathfrak{R}(z) = \{E \in \Gamma_m^2 : \ \pi[H(z)] \cap E \neq (0)\}$, if $r(z) = 2$,
- $\mathfrak{R}(z) = \{E \in \Gamma_m^2 : \ E \subset \pi[H(z)]\}$, if $r(z) = 3$ or 4;
- $\mathfrak{R}(z)$ is empty, if $r(z) > 4$.

Denote by $\mathfrak{R}$ the set-theoretic union of $\mathfrak{R}(z)$ over all $z \in \operatorname{Ker} \pi$. As is not hard to see, the decomposable elements in P are precisely the elements of rank ≤ 2. From this fact, taking Lemma 9 into account, it follows that

Lemma 11. Let $A = R_2$, $I, J \in \mathfrak{R}$, $I \neq J$, and let $E \in \Gamma_m^2$ be a two-dimensional subspace passing through the lines $\omega(I)$ and $\omega(J)$. In order that the ideals I and J form a proper pair, it is sufficient that $E \in \mathfrak{R}$.

Lemma 12. Let in the algebra R one have $A = R_2$. In order that there exist in S_A a connected absorbing set of marked planes, it is sufficient that the complement of $\mathfrak{R}$ in the manifold Γ_m^2 contain an interior point.

A situation in which the conditions of Lemma 12 are certainly fulfilled is easily determined from dimensional considerations. Indeed, the set $\mathfrak{R}(z)$ for $r(z) = 1, 3, 4$ and the set $\mathfrak{R}(z) \setminus \{H(z)\}$ for $r(z) = 2$ are submanifolds in Γ_m^2, whose (complex) dimensions are easy to estimate. At the same time, the dimension of the manifold Γ_m^2 itself, and also of the space P^2, is well known.

Theorem 3. Let R be a commutative complex algebra with m generators, in which the product of any three elements is equal to zero. Let

$$\dim R > \tfrac{1}{2} m(m+1) + 2.$$

Then the algebra R is uniquely determined, up to isomorphism, by its associated structure.

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2. D. A. Suprunenko, *UMN*, **11**, no. 3 (1956).

Note: Figure translations are in progress. See original paper for figures.

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