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MATHEMATICS

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Abstract

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MATHEMATICS

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ON A METHOD FOR OBTAINING INTERPOLATION THEOREMS IN SYMMETRIC SPACES

(Presented by Academician A. N. Kolmogorov on 29 VIII 1968)

Following ([1]), a Banach space E of functions measurable on $[0,1]$ is called symmetric if: 1) from $y(t) \in E$ and $|x(t)| \leq |y(t)|$ it follows that $x(t) \in E$ and $\|x\|_E \leq \|y\|_E$; 2) from $y(t) \in E$ and the equimeasurability of the functions $|x(t)|$ and $|y(t)|$ it follows that $x(t) \in E$ and $\|x\|_E = \|y\|_E$.

Many properties of a symmetric space E are characterized by the so-called fundamental function of the space E

$$\varphi_E(\tau) = \|\chi_{[0,\tau]}\|_E,$$

where $\chi_e(t)$ is the characteristic function of a measurable subset $e \subset [0,1]$.

In the class of symmetric spaces, important properties are possessed by the spaces $\Lambda(\psi)$, studied by G. G. Lorentz ([2]). Let $\psi(t)$ be an increasing concave function on $[0,1]$, $\psi(0) = 0$. Denote by $\Lambda(\psi)$ the set of all functions measurable on $[0,1]$ for which

$$\|x\|_{\Lambda(\psi)} = \int_0^1 x^*(t)\, d\psi(t) < \infty,$$

where $x^*(t)$ is a nonincreasing function equimeasurable with $|x(t)|$.

1°. The essence of the interpolation method for symmetric spaces, which will be described below, is as follows. For a symmetric space E, which is separable or conjugate to a separable space (everywhere in what follows we shall assume this requirement to be satisfied), there exists a set Ψ of increasing concave functions vanishing at 0 such that

$$\|x\|_E = \sup_{\psi \in \Psi} \|x\|_{\Lambda(\psi)}. \tag{1}$$

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Therefore, having certain information about the boundedness of a linear operator in some pairs of spaces of the type $\Lambda(\psi)$, one can, generally speaking, on the basis of (1) obtain theorems on the boundedness of the operator in arbitrary symmetric spaces. Here an important role is played by a theorem from ([3]), showing that under certain narrowings of the set Ψ one obtains new norms equivalent to the original ones. The proof of boundedness of a linear operator acting from the space $\Lambda(\psi)$ usually uses the structure of the set of extreme points of the unit ball of the space $\Lambda(\psi)$ ([1]).

We now give three concrete realizations of the interpolation method.

2°. Let $0 \le \beta_i \le \alpha_i \le 1$ $(i = 0, 1)$, $\alpha_0 \ne \alpha_1$, $\beta_0 \ne \beta_1$, and

$$\mu = (\beta_0 - \beta_1)/(\alpha_0 - \alpha_1), \quad \nu = (\alpha_0\beta_1 - \alpha_1\beta_0)/(\alpha_0 - \alpha_1).$$

If E is a given symmetric space, then denote by $E_{\mu,\nu}$ the set of all measurable functions for which

$$\|x^*(t^\mu)t^\nu\|_E < \infty.$$

Generally speaking, the triangle inequality is not satisfied for the functional $\|x^*(t^\mu)t^\nu\|_E$; however, in the case of interest to us, on $E_{\mu,\nu}$ one can define a norm $\|x\|_{E_{\mu,\nu}}$ equivalent to the functional $\|x^*(t^\mu)t^\nu\|_E$ and making $E_{\mu,\nu}$ into a Banach space.

Theorem 1. *Let a linear operator A be bounded from $\mathcal{L}_{1/\alpha_i}$ into $\mathcal{L}_{1/\beta_i}$ $(i = 0, 1)$. If*

$$2^{\alpha_0} < \lim_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)}, \qquad \overline{\lim_{t\to 0}} \frac{\varphi_E(2t)}{\varphi_E(t)} < 2^{\alpha_1},$$

then A is bounded from E into $E_{\mu,\nu}$.

We note that, instead of the continuity of A from $\mathcal{L}_{1/\alpha_i}$ into $\mathcal{L}_{1/\beta_i}$ $(i = 0, 1)$, one may require the fulfillment of either condition 1 or 2 (2 is weaker than 1):

1.
$$\tau\big(m\{t : x_e(t) > \tau\}\big)^{\beta_i} \le C(me)^{\alpha_i},$$

where C is independent of $\tau > 0$, $e \subset [0, 1]$, and $i = 0, 1$.

2. For every $s \in (0, 1)$

$$\|A\|_{\mathcal{L}_{1/\alpha_s} \to \mathcal{L}_{1/\beta_s}} < \infty,$$

where $\alpha_s = (1 - s)\alpha_0 + s\alpha_1$, $\beta_s = (1 - s)\beta_0 + s\beta_1$.

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Theorem 1 contains as special cases the interpolation theorems of J. Marcinkiewicz [4], E. M. Stein–G. Weiss [5], V. A. Dikarev–V. I. Matsaev [6], and preceding results of the author [7,8].

3°. Following N. Aronszajn and E. Gagliardo, a Banach space F is called an interpolation space with respect to a pair of Banach spaces F_0, F_1 if every linear operator A, continuous in F_0 and F_1, is continuous in F and

$$\|A\|_{F\to F} \le C \max_{i=0,1} \|A\|_{F_i\to F_i},$$

where C is independent of A.

It is not difficult to show that the following is true.

Theorem 2. *If E is an interpolation space with respect to the symmetric spaces E_0 and E_1, then, up to equivalence, E is symmetric.*

This assertion shows that, when considering interpolation spaces with respect to symmetric spaces, one may restrict oneself to symmetric spaces. Thus, let E_0, E, E_1 be symmetric spaces, and suppose that

$$\lim_{t\to 0} \frac{\varphi_{E_i}(2t)}{\varphi_{E_i}(t)}, \qquad \lim_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)}$$

exist and are equal to α_i $(i = 0, 1)$ and α, respectively, $\alpha_0 \le \alpha_1$.

Theorem 3. *The condition*

$$\alpha_0 \le \alpha \le \alpha_1$$

is necessary, and the condition

$$\alpha_0 < \alpha < \alpha_1$$

is sufficient, for E to be an interpolation space with respect to E_0 and E_1.

It can be shown that every symmetric Hilbert space coincides with $\mathcal{L}_2$, and the norm in such a space differs from $\|x\|_{\mathcal{L}_2}$ only by a factor. From this assertion and Theorem 3 it follows immediately:

Theorem 4. *The condition*

$$\alpha_0 \le \sqrt{2} \le \alpha_1$$

is necessary, and the condition

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$$\alpha_0 < \sqrt{2} < \alpha_1$$

is sufficient, for there to exist a Hilbert interpolation space with respect to E_0 *and* E_1.

4°. The convolution operator

$$x * y(t) = \int_0^1 x(t-s)y(s)\,ds$$

(on the interval $[-1, 0)$ the function $x(t)$ is extended periodically or is assumed to be equal to 0) can be regarded as a bilinear operator in spaces of functions defined on $[0, 1]$.

Theorem 5. Let E, F be symmetric spaces and suppose that there exists

$$\lim_{t\to 0} \frac{\varphi_F(2t)}{\varphi_F(t)}.$$

If

$$2 < \lim_{t\to 0} \frac{\varphi_F(2t)}{\varphi_F(t)} \lim_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)}, \qquad \overline{\lim}_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)} < 2,$$

then the convolution operator is continuous from $E \times F$ into G, where G is the space of all functions for which

$$\|x\|_G = \left\| x^*(t) \frac{\varphi_F(t)}{t} \right\|_E < \infty.$$

5°. In the theory of Banach spaces, the question of under what conditions a given Banach space has an unconditional basis is very important. For symmetric spaces this problem can be solved completely. With the aid of one result of A. M. Olevskii ([9]) and Theorem 1, one proves

Theorem 6. In order that there exist an unconditional basis in a symmetric space, it is necessary and sufficient that E be separable and

$$1 < \lim_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)}, \qquad \overline{\lim}_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)} < 2.$$

In the case when E is an Orlicz space, this assertion was obtained by V. F. Gaposhkin ([10]).

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Theorem 7. In order that the trigonometric system be a basis in a symmetric space E, it is necessary and sufficient that E be separable and

$$1 < \lim_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)}, \qquad \overline{\lim}_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)} < 2.$$

6°. In solving certain concrete problems, the concept of a symmetric space turns out to be too narrow. Therefore it is of interest to extend the concept of a symmetric space and the interpolation method to a broader class of spaces.

Let a measure m be defined on a set M. We shall assume that either m is continuous, or M is the union of a countable number of atoms of equal measure. Denote by $E = E(M, m)$ the set of measurable functions on M. The concept of a symmetric space of functions on M with respect to the measure m is introduced in exactly the same way as above. In a natural way the concept of a semiorderedness is introduced in $E(M, m)$.

Let now C be a linear set on which an operator P is defined with values in the sum of the spaces $\mathcal{L}_1(M, m)$, $\mathcal{L}_\infty(M, m)$, such that:

1) $P(x + y) \le Px + Py$ for all $x, y \in C$;
2) $Px \ge 0$ for all $x \in C$;
3) $P(\lambda x) = |\lambda| Px$;
4) $Px = 0 \Longleftrightarrow x = 0$.

It is easy to see that the functional $\|Px\|_E$ defined on C has all the properties of a norm. We shall assume that C is complete with respect to the norm

$$\|x\|_C = \|Px\|_E.$$

In this case the space $C = C(M, m, E, P)$ will be called symmetrized.

The interpolation method developed above for symmetric spaces carries over completely to symmetrized ones. As an example, we formulate an analogue of the most important Theorem 1.

Theorem 1′. Let the linear operator A be continuous from $C(M, m, \mathcal{L}_{1/\alpha_i}, P)$ to $C(M, m, \mathcal{L}_{1/\beta_i}, P)$. If

$$2^{\alpha_0} < \lim_{t\to 0} \frac{\varphi_E(2t)}{\varphi_E(t)}, \qquad \overline{\lim_{t\to 0}} \frac{\varphi_E(2t)}{\varphi_E(t)} < 2^{\alpha_1},$$

then A is continuous from $C(M, m, E, P)$ to $C(M, m, E_{\mu,\nu}, P)$.

We note that the constants in Theorems 1 and 1′ are sharp, i.e., the theorem ceases to be valid if 2^{α_0} is replaced by $2^{\alpha_0} - \varepsilon$ or 2^{α_1} by $2^{\alpha_1} + \varepsilon$. The same may be said of the constants in Theorems 5–7.

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