A COLLECTING PROCESS

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Abstract

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By a basic commutator we shall mean the basic commutators of Shirshov (see ([1], [2])). As is known, the Hall collecting process uses the following formulas (see ([3]), p. 188):

$$vu = uv[v, u], \tag{1}$$

$$v^{-1}u = u[v, u]^{-1}v^{-1}, \tag{2}$$

$$vu^{-1} \equiv u^{-1}uv_2 \cdots v_5^{-1}v_3^{-1}v_1^{-1} \pmod{\gamma_{k+1}(F)}, \tag{3}$$

$$v^{-1}u^{-1} \equiv u^{-1}uv_3 \cdots v_4^{-1}v_2^{-1}v^{-1} \pmod{\gamma_{k+1}(F)}, \tag{4}$$

where $\gamma_{k+1}(F)$ is the $(k+1)$-st term of the lower central series of the group $F = \gamma_1(F)$; $v_0 = v,\ v_{t+1} = [v_t, u]$. These formulas allow one, modulo $\gamma_{k+1}(F)$, to move to the left any commutator to a power ± 1; thus, if f is an arbitrary element of the free group F, then modulo any $\gamma_{k+1}(F)$ the element f can be represented uniquely in any base of basic commutators as

$$f = c_{i_1}^{\alpha_1} c_{i_2}^{\alpha_2} \cdots c_{i_n}^{\alpha_n} \pmod{\gamma_{k+1}(F)},$$

where the α_i are nonzero integers; c_{ij} are basic commutators, whose ordering in the base coincides with the ordering of their indices, $i_1 < i_2 < \cdots < i_n$ (see ([1]-[3])). Such a form of representation is called collected. One of the main merits of the Hall collecting process consists precisely in the statement just formulated. Another important type of collecting process is the Shirshov process. It was introduced by A. I. Shirshov (see ([1])) for Lie algebras and for groups by the authors in ([2]). In principle, this process makes it possible to do the same as the Hall process, but the use, instead of formulas (3), (4), of the Jacobi formula (see ([2])) makes it possible to determine better the form of the commutators that are obtained from a given commutator by means of the collecting process. The aim of this note is to get rid of the restrictive condition that the process is carried out modulo $\gamma_{k+1}(F)$, and, possibly, to reduce the number of commutators in the collecting process, i.e., to make the process faster.

Suppose that the ordering of the indices of the commutators of the base coincides with the ordering of the basic commutators. The associative word

$$c_{i_1} \cdots c_{i_m} c_{i_{m+1}}^{\varepsilon_1} \cdots c_{i_{m+k}}^{\varepsilon_k} c_{i_{m+k+1}}^{-1} \cdots c_{i_n}^{-1},$$

where c_{i_s} are basic commutators and $\varepsilon_j = \pm 1$ $(j = 1, \dots, k)$, contains, generally speaking, a positive collected part

$$c_{i_1} \cdots c_{i_m},$$

where $i_1 \leq \cdots \leq i_m$ $(i_m \leq i_j,\ j = m+1, \dots, m+k)$, and a negative collected part

$$c_{i_{m+k+1}}^{-1} \cdots c_{i_n}^{-1},$$

where $i_{m+k+1} \geq \cdots \geq i_n$ $(i_{m+k+1} \leq i_j,\ j = m+1, \dots, m+k)$, and an uncollected part

$$c_{i_{m+1}}^{\varepsilon_1} \cdots c_{i_{m+k}}^{\varepsilon_k},$$

where, if $\varepsilon_1 = 1$, i_{m+1} is not the smallest of the indices i_j $(j = m+1, \dots, m+k)$, and where, if $\varepsilon_k = -1$, i_{m+k} is not the smallest among the same indices. If the uncollected part is empty, then we shall say that the word is collected, or that it has collected form. The two-sided collecting process modulo N consists in the fact that commutators with the least index in the uncollected part of the word are shifted—first commutators with exponent $+1$ to the left, to the positive part, and then commutators with exponent -1 to the right, to the negative collected part; commutators that belong to N are discarded; for the shifts the following formulas are used:
for a shift to the left

$$uv = vu[u, v], \tag{5}$$

$$u^{-1}v = v[u, v]^{-1}u^{-1}, \tag{6}$$

for a shift to the left

$$u^{-1}v = v[v,u]u^{-1}, \tag{7}$$

$$v^{-1}u^{-1} = [v,u]^{-1}u^{-1}v^{-1}. \tag{8}$$

For the formulation of the results we need the following definitions: we shall call a basis of basic commutators natural modulo N if the set of commutators of this basis not belonging to N and composed of any finite number of symbols is ordered according to the type of the natural numbers, and is layerwise natural modulo the same N if before any commutator from $\rho_i(F) - \rho_{i+1}(F)$ $(i = 1, 2, \ldots)$, where $\rho_i(F)$ is the i-th term of the lower central series of the group F and $\rho_1(F) = F$, there exists only a finite number of basis commutators not belonging to $N \cup (\rho_i(F) - \rho_{i+1}(F))$. If $N = \langle 1 \rangle$, we shall omit the words "modulo N." Obviously, every natural basis is layerwise natural. An example of a natural basis is the Hall basis, consisting of commutators ordered by length. Examples of layerwise natural bases were constructed in the work of Bokut' ([4]); in particular, such is a basis of a free group in which a commutator from $\rho_i(F) - \rho_{i+1}(F)$ is smaller than a commutator from $\rho_j(F) - \rho_{j+1}(F)$ when $i < j$.

Theorem 1. Let

$$f = x_{i_1}^{\varepsilon_1} \cdots x_{i_s}^{\varepsilon_s}$$

be a word in the alphabet $x_1, x_2, \ldots, x_n$, where $\varepsilon_i = \pm 1$, and let m be a fixed natural number. Then, modulo $N \supseteq \rho_{m+1}(F)$ (N a normal subgroup of F), the word f can be brought by the two-sided collecting process to collected form in any layerwise natural modulo N basis.

Theorem 2. Any word can be brought by the two-sided collecting process modulo $\langle 1 \rangle$, in any layerwise natural basis, to a form whose uncollected part consists of commutators belonging to $\rho_{m+1}(F)$ for any fixed natural number m.

In other words, it is immaterial what is done first—to reduce modulo $\langle 1 \rangle$ and then discard commutators belonging to $\rho_{m+1}(F)$, or to discard such commutators immediately as they are obtained in the collecting process.

The situation with uniqueness of the collected form in the two-sided collecting process is poor. For example, consider the free group $F = \langle x, y \rangle$ and put $x < y$. As a basis take the Hall basis of commutators ordered by length. The word $f = x^{-1}y^{-1}xy$ can also be written as follows: $x^{-1}y^{-1}xx^{-1}xy$. Now apply the two-sided process to both forms of writing the word f:

$$f = x^{-1}y^{-1}xy = xy[y,x,x,y]^{-1}[y,x,y]^{-1} \times \\ \times [y,x,x]^{-1}[y,x]^{-1}y^{-1}x^{-1} \mod \rho_3(F),$$

$$f = x^{-1}y^{-1}xx^{-1}xy = xy[y,x,x,x,y]^{-1}[y,x,x,y]^{-2}[y,x,x,x]^{-1} \times$$

$$\times[y,x,y]^{-1}[y,x,x]^{-2}[y,x]^{-1}y^{-1}x^{-1}x^{-1} \pmod{\rho_3(F)}.$$

In the Hall process this does not happen: there the collected form of a word does not depend on the initial writing.

Before giving applications of Theorems 1 and 2, we introduce definitions. A subgroup N of a free group F is called commutatorial with respect to a certain basis if, for any natural number s, the factor groups $N\gamma_s(F)/\gamma_s(F)$ are generated by basic commutators from N, or, equivalently, if for every s all factor groups

$$(\gamma_s(F)\cap\gamma_{s+1}(F)N)/\gamma_{s+1}(F)$$

are generated by basic commutators from N. A Malcev group (or Mal' cev group) will mean a group G satisfying the conditions:

1) $\bigcap_n \gamma_n(G) = 1, \qquad n$ is a natural number;
2) $\gamma_n(G)/\gamma_{n+1}(G)$ is a free abelian group.

Theorem 3. *If a normal subgroup N of a free group F, commutatorial in a basis that is natural layer by layer modulo $\rho_k(F)$, contains $\rho_k(F)$, then the group F/N is Malcev.*

Theorem 4. *If all the groups $N\rho_n(F)$, where N is a normal subgroup of the free group F and n is a natural number, are commutatorial in some layerwise natural basis, then from the approximability of the group F/N by solvable groups it follows that F/N is a Malcev group.*

The cases where N is a verbal subgroup of the free group F defining the free group of a variety of solvable, polynilpotent, multinilpotent, and other groups were considered, for example, in papers ([2], [5–8]).

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Note: Figure translations are in progress. See original paper for figures.